Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations

Stefan Le Coz, Dong Li, Tai-Peng Tsai

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Abstract

We study *infinite soliton trains* solutions of nonlinear Schrödinger equations (NLS), i.e. solutions behaving at large time as the sum of infinitely many solitary waves. Assuming the composing solitons have sufficiently large relative speeds, we prove the existence and uniqueness of such a soliton train. We also give a new construction of multi-solitons (i.e. finite trains) and prove uniqueness in an exponentially small neighborhood, and we consider the case of solutions composed of several solitons and kinks (i.e. solutions with a non-zero background at infinity).

Keywords: soliton train, multi-soliton, multi-kink, nonlinear Schrödinger equations.

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1 Introduction

We consider the following nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u = -g(|u|^2)u =: -f(u), \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$, $d \geq 1$.

The purpose of this paper is to construct special families of solutions to the energy-subcritical NLS (1.1). We will look for *infinite soliton trains, multi-solitons* and *multi-kinks* solutions.

Recall that it is generically expected that global solutions to nonlinear dispersive equations like NLS eventually decompose at large time as a sum of solitons plus a scattering remainder (*Soliton Resolution Conjecture*). Except for the specific case of integrable equations, such results are usually out of reach (see nevertheless the recent breakthrough on energy-critical wave equation [10]). In the case of nonlinear Schrödinger equations, multi-solitons can be constructed via the inverse scattering transform in the integrable case ($d = 1$, $f(u) = |u|^2u$). In non-integrable frameworks, multi-solitons are known to exist since the pioneering work of Merle [21] (see Section 1.2 for more details on the existing results of multi-solitons). The multi-solitons constructed up to now were made of a finite number of

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solitons and there was little evidence of the possibility of existence of infinite trains of solitons (note nevertheless the result [13] in the integrable case). The existence of such infinite solitons trains is however important, as they may provide examples or counter-examples of solutions with borderline behaviors (as it is the case for the Korteweg-de Vries equation, see [17]). In this paper, we show the existence of such infinite soliton trains for power nonlinearities. It turns out that our strategy is very flexible and allows us to prove many results of existence and uniqueness of multi-solitons and multi-kinks solutions for generic nonlinearities. In the rest of this introduction, we state our main results on infinite trains (Section 1.1), multi-solitons (Section 1.2) and multi-kinks (Section 1.3) and give a summary of the strategy of the proofs (Section 1.4).

1.1 Infinite soliton trains

Our first main result is on the construction of a solution to (1.1) behaving at large time like a sum of infinitly many solitons. For this purpose we have to use scale invariance and work with the power nonlinearity $f_1(u) = |u|^\alpha u$, $0 < \alpha < \alpha_{\max}$, $\alpha_{\max} = +\infty$ for $d = 1, 2$ and $\alpha_{\max} = \frac{4}{d-2}$ for $d \geq 3$. Let $\Phi_0 \in H^1(\mathbb{R}^d)$ be a fixed bound state which solves the elliptic equation

$$-\Delta\Phi_0 + \Phi_0 - |\Phi_0|^\alpha\Phi_0 = 0.$$

For $j \geq 1$, $\omega_j > 0$ (*frequency*), $\gamma_j \in \mathbb{R}$ (*phase*), $v_j \in \mathbb{R}^d$ (*velocity*), define a soliton $\tilde{R}_j$ by

$$\tilde{R}_j(t,x) := e^{i(\omega_j t - \frac{|v_j|^2}{4} + \frac{1}{2}v_j\cdot x + \gamma_j)}\omega_j^{\frac{1}{\alpha}}\Phi_0\big(\sqrt{\omega_j}(x - v_j t)\big). \tag{1.2}$$

We consider the following soliton train:

$$R_\infty = \sum_{j=1}^{\infty} \tilde{R}_j. \tag{1.3}$$

Since (1.1) is a nonlinear problem, the function $R_\infty = R_\infty(t,x)$ is no longer a solution in general. Nevertheless we are going to show that in the vicinity of R_∞ one can still find a solution u to (1.1) to which we refer to as an *infinite soliton train*. More precisely, the solution u to (1.1) is defined on $[T_0, +\infty)$ for some $T_0 \in \mathbb{R}$ and such that

$$\lim_{t\to+\infty} \|u - R_\infty\|_{X([t,\infty)\times\mathbb{R}^d)} = 0. \tag{1.4}$$

Here $\|\cdot\|_{X([t,\infty)\times\mathbb{R}^d)}$ is some space-time norm measured on the slab $[t,\infty)\times\mathbb{R}^d$. A simple example is $X = L_t^\infty L_x^2$ in which case one can replace (1.4) by the equivalent condition

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{L^2} = 0.$$

However the definition (1.4) is more flexible as one can allow general Strichartz spaces (see (2.2)).

The main idea is that in the energy-subcritical setting, all solitons have exponential tails (see (1.13)). When their relative speed is large, these traveling solitons are well-separated and have very small overlaps which decay exponentially in time. At such high velocity and exponential separation, one does not need fine spectral details and the whole argument can be carried out as a perturbation around the desired profile (e.g. the soliton series R) in a well-chosen function space. As our proof is based on contraction estimates, the uniqueness follows immediately, albeit in a very restrictive function class.

We require that the parameters (ω_j, v_j) of the train satisfy the following assumption.

Assumption A.

- *(Integrability) There exists $r_1 \geq 1$, $\frac{d\alpha}{2} < r_1 < \alpha + 2$, such that*

$$A_\omega := \sum_{j=1}^{\infty} \omega_j^{\frac{1}{\alpha} - \frac{d}{2r_1}} < \infty. \tag{1.5}$$

- *(High relative speeds) The solitons travel sufficiently fast: there exists a constant $v_\star > 0$ such that*

$$\sqrt{\min\{\omega_j, \omega_k\}} \Big(|v_k - v_j| \Big) \geq v_\star, \qquad \forall\, j \neq k. \tag{1.6}$$

Since R_∞ may be badly localized, we seek a infinite soliton train solution to (1.1) in the form $u = R_\infty + \eta$, where η satisfies the perturbation equation

$$i\partial_t \eta + \Delta \eta = -f(R_\infty + \eta) + \sum_{j=1}^{\infty} f(\tilde{R}_j).$$

In Duhamel formulation, the perturbation equation for η reads

$$\eta(t) = i \int_t^{\infty} e^{i(t-\tau)\Delta} \Big(f(R_\infty + \eta) - \sum_{j=1}^{\infty} f(\tilde{R}_j) \Big) d\tau, \qquad \forall\, t \geq 0. \tag{1.7}$$

The following theorem gives the existence and uniqueness of the solution η to (1.7).

Theorem 1.1 (Existence of an infinite soliton train solution)**.** *Consider* (1.1) *with* $f(u) = |u|^\alpha u$ *satisfying* $0 < \alpha < \alpha_{\max}$*. Let* R_∞ *be given as in* (1.3)*, with parameters* $\omega_j > 0$*,* $\gamma_j \in \mathbb{R}$*, and* $v_j \in \mathbb{R}^d$ *for* $j \in \mathbb{N}$*, which satisfy Assumption A. There exist constants* $C > 0$*,* $c_1 > 0$ *and* $v_\sharp \gg 1$ *such that* (*see* (1.6)) *if* $v_\star > v_\sharp$*, then there exists a unique solution* $\eta \in S([0,\infty))$ *(see* (2.2) *for the definition of Strichartz space) to* (1.7) *satisfying*

$$\|\eta\|_{S([t,\infty))} + \|\eta(t)\|_{L^{\alpha+2}} \leq C e^{-c_1 v_\star t}, \qquad \forall\, t \geq 0. \tag{1.8}$$

Remark 1.2. By using Theorem 1.1 and Lemma 4.1, one can justify the existence of a solution $u = R_\infty + \eta$ satisfying (1.1) in the distributional sense. The uniqueness of such solutions is only proven for the perturbation η satisfying (1.7) and (1.8). In the mass-subcritical case $0 < \alpha < \frac{4}{d}$, the soliton train R_∞ is in the Lebesgue space $C_t^0 L_x^2 \cap L_{tx}^\infty$, and one can show that the solution $u = R_\infty + \eta$ can be extended to all $\mathbb{R} \times \mathbb{R}^d$ and satisfies $u \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^d) \cap L_{t,loc}^{\frac{2(d+2)}{d}} L_x^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d)$ (see (2.1)). Hence it is a localized solution in the usual sense. In the mass-supercritical case $\frac{4}{d} \leq \alpha < \alpha_{\max}$, the soliton train $R_\infty = \sum_{j=1}^{\infty} \tilde{R}_j$ is no longer in L^2 since each composing piece $\tilde{R}_j$ has $O(1)$ L^2-norm. Nevertheless we shall still build a regular solution to (1.7) since R_∞ has Lebesgue regularity $L_t^\infty L_x^{\frac{d\alpha}{2}+} \cap L_{tx}^\infty$ which is enough for the perturbation argument to work. We stress that in this case the solution η is only defined on $[0,\infty) \times \mathbb{R}^d$ and scatters forward in time in L^2.

Remark 1.3. Typically the parameters (ω_j, v_j) are chosen in the following order: first we take (ω_j) satisfying (1.5); then we inductively choose v_j such that the condition (1.6) is satisfied. For example one can take for $j \geq 1$, $\omega_j = 2^{-j}$ and $v_j = 2^j \bar{v}$ for $\bar{v} \in \mathbb{R}^d$, $|\bar{v}| = v_\star$. Note that, when $0 < \alpha < \frac{4}{d}$ (mass-subcritical case), we can choose $r_1 \leq 2$. The soliton train is then in $L_t^\infty L_x^2$. We require $\frac{d\alpha}{2} < r_1$ so that the exponent in (1.5) is positive. The condition $r_1 < \alpha + 2$ will be needed to show (4.2) in Lemma 4.1.

Remark 1.4. Note that we did not introduce initial positions in the definition of $\tilde{R}_j$, so each soliton starts centered at 0. With some minor modifications, our construction can also work for the general case with the solitons starting centered at various x_j. For simplicity of presentation we shall not state the general case here.

Remark 1.5. Certainly Theorem 1.1 can hold in more general situations. For example instead of taking a fixed profile Φ_0 in (1.2), one can draw Φ_0 from a finite set of profiles $\mathcal{A} = \{\Phi_0^1, \cdots, \Phi_0^K\}$ where each Φ_0^j is a bound state.

Remark 1.6. The rate of spatial decay of multi-solitons is still an open question in the NLS case (for KdV it is partly known: multi-solitons decay exponentially on the right). In Theorem 1.1, the soliton train profile R_∞ around which we build our solution has only a polynomial spatial decay, not uniform in time. Hence we expect the solution $u = R_\infty + \eta$ to have the same decay.

1.2 Multi-solitons

From now on, we work with a generic nonlinearity and just assume that $f(u) = g(|u|^2)u$ where the function $g : [0, \infty) \to \mathbb{R}$ obeys some Hölder conditions mimicking the usual power type nonlinearity. Precisely,

- $g \in C^0([0,\infty), \mathbb{R}) \cap C^2((0,\infty), \mathbb{R})$, $g(0) = 0$ and

$$|sg'(s)| + |s^2 g''(s)| \leq C \cdot (s^{\alpha_1} + s^{\alpha_2}), \qquad \forall\, s > 0, \tag{1.9}$$

  where $C > 0$, $0 < \alpha_1 \leq \alpha_2 < \frac{\alpha_{\max}}{2}$.

A typical example is $g(s) = s^\alpha$ for some $0 < \alpha < \frac{\alpha_{\max}}{2}$. A useful example to keep in mind is the combined nonlinearity $g(s) = s^{\alpha_1} - s^{\alpha_2}$ for some $0 < \alpha_1 < \alpha_2 < \frac{\alpha_{\max}}{2}$. Other examples can be easily constructed. Throughout the rest of this paper we shall assume $f(u) = g(|u|^2)u$ satisfy (1.9). In this case the corresponding nonlinearity $f(u)$ is usually called energy-subcritical since there are lower bounds of the lifespans of the H^1 local solutions which depend only on the H^1-norm (not the profile) of initial data (cf. [5, 11]). The condition (1.9) is a natural generalization of the pure power nonlinearities. For much of our analysis it can be replaced by the weaker condition that $g(s)$ and $sg'(s)$ are Hölder continuous with suitable exponents. However (1.9) is fairly easy to check and it suffices for most applications.

We give a definition of a solitary wave slightly more general than (1.2). Given a set of parameters $\omega_0 > 0$ (*frequency*), $\gamma_0 \in \mathbb{R}$ (*phase*), x_0, $v_0 \in \mathbb{R}^d$ (*position* and *velocity*), a *solitary wave*, or a *soliton*, is a solution to (1.1) of the form

$$R_{\Phi_0,\omega_0,\gamma_0,x_0,v_0} := \Phi_0(x - v_0 t - x_0) \exp\left(i\Big(\frac{1}{2} v_0 \cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0\Big)\right), \tag{1.10}$$

where $\Phi_0 \in H^1(\mathbb{R}^d)$ solves the elliptic equation

$$-\Delta\Phi_0 + \omega_0\Phi_0 - f(\Phi_0) = 0. \tag{1.11}$$

A nontrivial H^1 solution to (1.11) is usually called a *bound state*. Compared with (1.2), the main difference is that we do not use the parameter ω_j to rescale the solitons.

Existence of bound states is guaranteed (see [1]) if we assume, in addition to (1.1), that there exists $s_0 > 0$, such that

$$G(s_0) := \int_0^{s_0} g(\tilde{s})d\tilde{s} > \omega_0 s_0. \tag{1.12}$$

Note that the condition (1.12) makes the nonlinearity focusing.

All bound states are exponentially decaying (cf. Section 3.3 of [3] for example), i.e.

$$e^{\sqrt{\omega}|x|}(|\Phi_0| + |\nabla\Phi_0|) \in L^\infty(\mathbb{R}^d), \quad \text{for all } 0 < \omega < \omega_0. \tag{1.13}$$

A *ground state* is a bound state which minimizes among all bound states the *action*

$$S(\Phi_0) = \frac{1}{2}\|\nabla\Phi_0\|_2^2 + \frac{\omega_0}{2}\|\Phi_0\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^d} G(|\Phi_0|^2)dx.$$

The ground state is usually unique modulo symmetries of the equation (see e.g. [20] for precise conditions on the nonlinearity ensuring uniqueness of the ground state). If $d \geq 2$ there exist infinitely many other solutions called *excited states* (see [1, 2] for more on ground states and excited states). The corresponding solitons are usually termed *ground state solitons* (resp. *excited state solitons*).

A *multi-soliton* is a solution to (1.1) which roughly speaking looks like the sum of N solitons. To fix notations, let (see (1.10))

$$R(t,x) = \sum_{j=1}^N R_{\Phi_j,\omega_j,\gamma_j,x_j,v_j}(t,x) =: \sum_{j=1}^N R_j(t,x), \tag{1.14}$$

where each R_j is a soliton made from some parameters $(\omega_j, \gamma_j, x_j, v_j)$ and bound state Φ_j (we assume that (1.12) holds true for all ω_j).

If each Φ_j in (1.14) is a ground state, then the corresponding multi-soliton is called a *ground state multi-soliton*. If at least one Φ_j is an excited state, we call it an *excited state multi-soliton*.

We now review in more details some known results on multi-solitons. Most results are on the pure power nonlinearity $f(u) = |u|^\alpha u$ with $0 < \alpha < \alpha_{\max}$ and ground states. If $\alpha = \frac{4}{d}$ (resp. $\alpha < \frac{4}{d}$, $\alpha > \frac{4}{d}$), then equation (1.1) is called (L^2) mass-critical (resp. mass-subcritical, mass-supercritical). In the integrable case $d = 1$, $\alpha = 2$, Zakharov and Shabat [25] derived an explicit expression of multi-solitons by using the inverse scattering transform. For the mass-critical NLS, which is non-integrable in higher dimensions, Merle [21] (see Corollary 3 therein) constructed a solution blowing up at exactly N points at the same time, which gives a multi-soliton after a pseudo-conformal transformation. In the mass-subcritical case, the ground state solitary waves are stable. Assuming the composing solitary waves R_j are ground states and have different velocities (i.e. $v_j \neq v_k$ if $j \neq k$ in (1.14)), Martel and Merle [18] proved the existence of an H^1 ground state multi-soliton $u \in C([T_0,\infty), H^1)$ such that

$$\Big\|u(t) - \sum_{j=1}^N R_j(t)\Big\|_{H^1} \leq Ce^{-\beta\sqrt{\omega_\star}v_\star t}, \qquad \forall\, t \geq T_0, \tag{1.15}$$

for some constant $\beta > 0$, where $T_0 \in \mathbb{R}$ is large enough, and the minimal relative velocity $v_\star$ and the minimal frequency $\omega_\star$ are defined by

$$v_\star := \min\{|v_j - v_k| : \ 1 \le j \ne k \le N\}, \tag{1.16}$$

$$\omega_\star = \min\{\omega_j, \ 1 \le j \le N\}. \tag{1.17}$$

In the same work, the authors also considered a general energy-subcritical nonlinearity $f(u) = g(|u|^2)u$ with $g \in C^1$, $g(0) = 0$ and satisfy $\|s^{-\alpha} g'(s)\|_{L^\infty_s(s\ge 1)} < \infty$ for some $0 < \alpha < \alpha_{\max}/2$. Assuming a nonlinear stability condition around the ground state (see (16) of [18]), they proved the existence of an H^1 ground state multi-soliton satisfying the same estimate (1.15).

In [9], Côte, Martel and Merle considered the mass-supercritical NLS ($f(u) = |u|^\alpha u$ with $\frac{4}{d} < \alpha < \alpha_{\max}$). Assuming the ground state solitons R_j have different velocities, the authors constructed an H^1 ground state multi-soliton u satisfying (1.15). This result was sharpened in 1D by Combet: in [7], he showed the existence of a N-parameters family of multi-solitons.

In [8], Côte and Le Coz considered the general energy-subcritical NLS with $f(u) = g(|u|^2)u$ satisfying assumptions similar to (1.9) and (1.12). Assuming the solitary waves R_j are excited states and have large relative velocities, i.e. assuming

$$v_\star \ge v_\sharp > 0$$

for $v_\sharp$ large enough, the authors constructed an excited state multi-soliton $u \in C([T_0, \infty), H^1)$ for $T_0 \in \mathbb{R}$ large enough, which also satisfies (1.15).

The main strategy used in the above mentioned works [8, 9, 18, 21] is the following: one takes a sequence of approximate solutions u_n solving (1.1) with final data $u_n(T_n) = R(T_n)$, $T_n \to \infty$; by using local conservation laws and coercivity of the Hessian (this has to be suitably modified in certain cases, cf. [8]), one derives uniform H^1 decay estimates of u_n on the time interval $[T_0, T_n]$ where T_0 is independent of n; the multi-soliton is then obtained after a compactness argument. We should point out that the uniqueness of multi-solitons is still left open by the above analysis (see nevertheless [7, 8] for existence of a 1 and N parameters families of multi-solitons). Under restrictive assumptions on the nonlinearity (e.g. high regularity or flatness assumption at the origin) and a large relative speeds hypothesis, stability of multi-solitons was obtained in [19, 22, 23, 24] and instability in [8]. See also Remark 1.11 below.

In this section we give new constructions of multi-solitons. We work in the context of the energy-subcritical problem (1.1) with $f(u)$ satisfying (1.9) and (1.12) We shall focus on *fast-moving* solitons, i.e. the minimum relative velocity $v_\star$ defined in (1.16) is sufficiently large. The composing solitons are in general bound states which can be either ground states or excited states. In our next two results, we recover and improve the result from [8, Theorem 1] in various settings. The improvements here are the lifespan and uniqueness. As for the infinite train, our new proof rely on a contraction argument around the desired profile. We begin with the pure power nonlinearity case.

Theorem 1.7 (Existence and uniqueness of multi-solitons, power nonlinearity case)**.** *Consider* (1.1) *with $f(u) = |u|^\alpha u$ satisfying $0 < \alpha < \alpha_{\max}$. Let R be the same as in* (1.14) *and define $v_\star$ as in* (1.16)*. There exists constants $C > 0$, $c_1 > 0$ and $v_\sharp \gg 1$ such that if $v_\star > v_\sharp$, then there exists a unique solution $u \in C([0, \infty), H^1)$ to* (1.1) *satisfying*

$$e^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + e^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \le C, \qquad \forall\, t \ge 0.$$

Here $c_2 = c_1 \cdot \min(1, \alpha) \leq c_1$. In particular $\|u(t) - R(t)\|_{H^1} \leq Ce^{-c_2 v_\star t}$.

Remark 1.8. As was already mentioned, Theorem 1.7 is a slight improvement of a corresponding result (Theorem 1) in [8]. Here the multi-soliton is constructed on the time interval $[0, \infty)$ whereas in [8] this was done on $[T_0, \infty)$ for some $T_0 > 0$ large. In particular, we do not have to wait for the interactions between the solitons to be small to have existence of our multi-soliton. However, we have no control on the constant C so at small times our multi-soliton may be very far away from the sum of solitons. The uniqueness of solutions is a subtle issue, see Remark 1.11.

The next result concerns the general nonlinearity $f(u)$.

Theorem 1.9 (Existence and uniqueness of multi-solitons, general nonlinearity case)**.** *Consider* (1.1) *with $f(u) = g(|u|^2)u$ satisfying* (1.9) *and* (1.12)*. Let R be the same as in* (1.14) *and define $v_\star$ as in* (1.16)*. There exist constants $C > 0$, $c_1 > 0$, $c_2 > 0$, $T_0 \gg 1$ and $v_\sharp \gg 1$, such that if $v_\star > v_\sharp$, then there is a unique solution $u \in C([T_0, \infty), H^1)$ to* (1.1) *satisfying*

$$e^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + e^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \leq C, \qquad \forall\, t \geq T_0.$$

Remark 1.10. Unlike Theorem 1.7, the solution in Theorem 1.9 exists only for $t \geq T_0$ with T_0 sufficiently large. To take $T_0 = 0$, our method requires extra conditions. For such results see Section 6. We can also extend Theorem 1.16 similarly.

Remark 1.11. In Theorems 1.7 and 1.9, the uniqueness of the multi-soliton solution holds in a quite restrictive function class whose Strichartz-norm decay as $e^{-c_1 v_\star t}$. A natural question is whether uniqueness holds in a wider setting. In general this is a very subtle issue and in some cases one cannot get away with the exponential decay condition. In [8], the authors considered the case when one of the composing soliton, say R_1 is unstable. Assuming $g \in C^\infty$ (see (1.1)) and the operator $L = -i\Delta + i\omega_1 - idf(\Phi_1)$ has an eigenvalue $\lambda_1 \in \mathbb{C}$ with $\rho := \mathrm{Re}(\lambda_1) > 0$, they constructed a one-parameter family of multi-solitons $u_a(t)$ such that for some $T_0 = T_0(a) > 0$,

$$\|u_a(t) - \sum_{j=1}^{N} R_j(t) - aY(t)\|_{H^1(\mathbb{R}^d)} \leq Ce^{-2\rho t}, \qquad \forall\, t \geq T_0.$$

Here $Y(t)$ is a nontrivial solution of the linearized flow around R_1, and $e^{\rho t}\|Y(t)\|_{H^1}$ is periodic in t. This instability result shows that the exponential decay condition in the uniqueness statement cannot be removed in general for NLS with unstable solitary waves.

1.3 Multi-kinks

In this subsection, we push our approach further and attack the problem of the existence of multi-kinks, i.e. solutions built upon solitons and their nonlocalized counterparts the kinks. Before stating our result, let us first mention some related works. When its solutions are considered with a non-zero background (i.e. $|u| \to \nu \neq 0$ at $\pm\infty$), the NLS equation (1.1) is often refered to as the Gross-Pitaevskii equation. For general non-linearities, Chiron [6] investigated the existence of traveling wave solutions with a non-zero background and showed that various types of nonlinearities can lead to a full zoology of profiles for the traveling waves. In the case of the "classical" Gross-Pitaevskii equation, i.e. when $f(u) =$

$(1-|u|^2)u$ and solutions verify $|u| \to 1$ at infinity, the profiles of the traveling kink solutions $K(t,x) = \phi_c(x-ct)$ are explicitly known and given for $|c| < \sqrt{2}$ by the formula

$$\phi_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(\frac{x\sqrt{2-c^2}}{2}\right) + i\frac{c}{\sqrt{2}}$$

with $\omega = 0$. (in particular, one can see that the limits at $-\infty$ and $+\infty$ are different, thus justifying the name "kink"). In [4], Béthuel, Gravejat and Smets proved the stability forward in time of a profile composed of several kinks traveling at different speeds. Note that, due to the non-zero background of the kinks, the profile cannot be simply taken as a sum of kinks and one has to rely on another formulation of the Gross-Pitaevskii equation to define properly what is a multi-kink.

The main differences between our analysis and the works above mentioned are, first, that our kinks have a zero background on one side and a non-zero one on the other side, and second, that, due to the Galilean transform used to give a speed to the kink, our kinks have infinite energy (due to the non-zero background, the rotation in phase generated by the Galilean transform is not killed any more by the decay of the modulus). In particular, this would prevent us to use energy methods as it was the case for multi-solitons in [8, 9, 18] or multi-kinks [4].

We place ourselves in dimension $d = 1$. In such context and under suitable assumptions on the nonlinearity f, (1.1) admits kink solutions. More precisely, given $\gamma, \omega, v, x_0 \in \mathbb{R}$, what we call a *kink* solution of (1.1) (or *half-kink*) is a function $K = K(t,x)$ defined similarly as a soliton by

$$K(t,x) := e^{i\left(\frac{1}{2}vx - \frac{1}{4}|v|^2 t + \omega t + \gamma\right)}\phi(x - vt - x_0),$$

but where ϕ satisfies the profile equation on $\mathbb{R}$ *with a non-zero boundary condition* at one side of the real line, denoted by $\pm\infty$ and zero boundary condition on the other side (denoted by $\mp\infty$):

$$\begin{cases} -\phi'' + \omega\phi - f(\phi) = 0, \\ \lim\limits_{x\to\mp\infty} \phi(x) = 0, \quad \lim\limits_{x\to\pm\infty} \phi(x) \neq 0. \end{cases} \tag{1.18}$$

The existence of half-kinks is granted by the following proposition.

Proposition 1.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(0) = 0$ and define $F(s) := \int_0^s f(t)dt$. For $\omega \in \mathbb{R}$, let*

$$\zeta(\omega) := \inf\{\zeta > 0, F(\zeta) - \frac{\omega}{2}\zeta^2 = 0\}$$

and assume that there exists $\omega_1 \in \mathbb{R}$ such that

$$\zeta(\omega_1) > 0, \qquad f'(0) - \omega_1 < 0, \qquad f(\zeta(\omega_1)) - \omega_1\zeta(\omega_1) = 0. \tag{1.19}$$

Then for $\omega = \omega_1$ there exists a kink profile solution $\phi \in \mathcal{C}^2(\mathbb{R})$ of (1.18), i.e. ϕ is unique (up to translation), positive and satisfies $\phi > 0$, $\phi' > 0$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty} \phi(x) = 0, \qquad \lim_{x\to+\infty} \phi(x) = \zeta(\omega_1) > 0. \tag{1.20}$$

If in addition

$$f'(\zeta(\omega_1)) - \omega_1 < 0,$$

then for any $0 < \delta < \omega_1 - \max\{f'(0), f'(\zeta(\omega_1))\}$ there exists $C > 0$ such that

$$|\phi'(x)| + |\phi(x)\mathbf{1}_{\{x<0\}}| + |(\zeta_1(\omega_1) - \phi(x))\mathbf{1}_{\{x>0\}}| \leq Ce^{-\delta|x|}. \tag{1.21}$$

Remark 1.13. By uniqueness we mean that when $\omega = \omega_1$ the only solutions connecting 0 to $\zeta(\omega_1)$ (i.e. satisfying (1.20)) are of the form $\phi(\cdot + c)$ for some $c \in \mathbb{R}$.

Remark 1.14. Using the symmetry $x \to -x$ it is easy to see that Proposition 1.12 also implies the existence and uniqueness of a kink solution ϕ satisfying

$$\lim_{x\to-\infty} \phi(x) = \zeta(\omega_1) > 0, \qquad \lim_{x\to+\infty} \phi(x) = 0.$$

Reverting the $>$ into the assumptions of Proposition 1.12 we immediately obtain the existence of a kink profile connecting 0 to $\zeta(\omega_1) < 0$.

Remark 1.15. It is well known (see [1]) that if instead of (1.19) we assume that there exists $\omega_0 \in \mathbb{R}$ such that

$$\zeta(\omega_0) > 0, \qquad f(\zeta(\omega_0)) - \omega\zeta(\omega_0) > 0,$$

then for $\omega = \omega_0$ there exists a soliton profile, i.e. a unique positive even solution $\phi \in \mathcal{C}^2(\mathbb{R})$ to (1.18) with boundary conditions

$$\lim_{x\to\pm\infty} \phi(x) = 0.$$

The profile on which we want to build a solution to (1.1) is the following. Take $N \in \mathbb{N}$, $(v_j, x_j, \omega_j, \gamma_j)_{j=0,\dots,N+1} \subset \mathbb{R}^4$ such that $v_0 < \cdots < v_{N+1}$. Assume that for ω_0 and ω_{N+1} there exist two kink profiles ϕ_0 and ϕ_{N+1} (solutions of (1.18)) satisfying the boundary conditions

$$\begin{aligned} &\lim_{x\to-\infty} \phi_0(x) \neq 0, && \lim_{x\to+\infty} \phi_0(x) = 0, \\ &\lim_{x\to-\infty} \phi_{N+1}(x) = 0, && \lim_{x\to+\infty} \phi_{N+1}(x) \neq 0. \end{aligned}$$

Denote by K_0 and K_{N+1} the corresponding kinks. For $j = 1, \dots, N$, assume as before that we are given localized solitons profiles $(\phi_j)_{j=1,\dots,N}$ and let R_j be the corresponding solitons. Consider the following approximate solution composed of a kink on the left and on the right and solitons in the middle (see Figure 1):

$$KR(t,x) := K_0(t,x) + \sum_{j=1}^{N} R_j(t,x) + K_{N+1}(t,x). \tag{1.22}$$

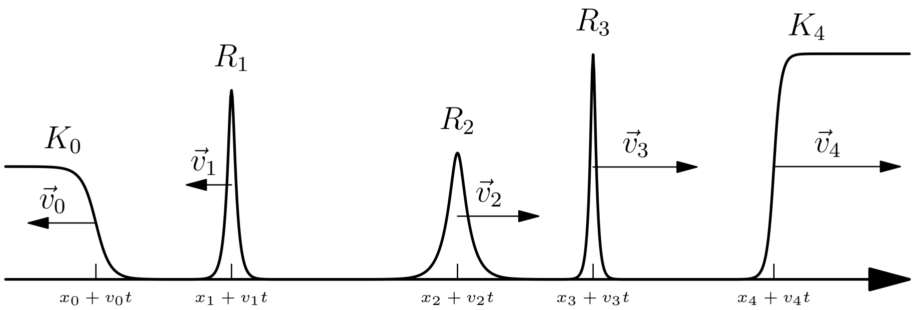

Figure 1: Schematic representation of the multi-kink profile KR in (1.22)

Our last result concerns solutions of that are composed of solitons and half-kinks.

Theorem 1.16. *Consider* (1.1) *with* $d = 1$, $f(u) = g(|u|^2)u$ *satisfying* (1.9), *and let* KR *be the profile defined in* (1.22). *Define* $v_\star$ *by*

$$v_\star := \inf\{|v_j - v_k|;\ j, k = 0, \dots, N+1,\ j \neq k\}.$$

Then there exist $v_\sharp > 0$ *(independent of* (v_j)*) large enough,* $T_0 \gg 1$ *and constants* $C, c_1, c_2 > 0$ *such that if* $v_\star > v_\sharp$, *then there exists a (unique) multi-kink solution* $u \in \mathcal{C}([T_0, +\infty), H^1_{\mathrm{loc}}(\mathbb{R}))$ *to* (1.1) *satisfying*

$$e^{c_1 v_\star t}\|u - KR\|_{S([t,+\infty))} + e^{c_2 v_\star t}\|\nabla(u - KR)\|_{S([t,+\infty))} \leq C, \qquad \forall t \geq T_0.$$

It will be clear from the proof that the theorem remains valid if we remove K_0 or K_{N+1} from the profile KR. It is also fine if $v_0 > 0$ or $v_{N+1} < 0$.

1.4 Strategy of the proofs

To simplify the presentation, we shall give a streamlined proof to Theorems 1.1, 1.7, 1.9 and 1.16. The key tools are Proposition 2.3 and Proposition 2.4 which reduce matters to the checking of a few conditions on the solitons. This is done in Section 2. We stress that the situation here is a bit different from the usual stability theory in critical NLS problems (cf. [15, 16]). There the approximate solutions often have finite space-time norms and the perturbation errors only need to be small in some dual Strichartz space. In our case the solitary waves carry infinite space-time norms on any non-compact time interval (unless one considers L^∞_t). For this we have to rework a bit the stability theory around a solitary wave type solution. The price to pay is that the perturbation errors and source terms need to be exponentially small in time. This is the main place where the large relative velocity assumption is used. We give the proofs of Theorems 1.7 and 1.9 in Section 3, of Theorem 1.1 in Section 4 and finally of Theorem 1.16 in Section 5. In Section 6, we conclude the paper by giving three results similar to Theorem 1.9 with additional assumptions that allow us to take $T_0 = 0$.

2 The perturbation argument

We start this section by giving some

Preliminaries and notations

For any two quantities A and B, we use $A \lesssim B$ (resp. $A \gtrsim B$) to denote the inequality $A \leq CB$ (resp. $A \geq CB$) for a generic positive constant C. The dependence of C on other parameters or constants is usually clear from the context and we will often suppress this dependence. Sometimes we will write $A \lesssim_k B$ if the implied constant C depends on the parameter k. We shall use the notation $C = C(X)$ if the constant C depends explicitly on some quantity X.

For any function $f : \mathbb{R}^d \to \mathbb{C}$, we use $\|f\|_{L^p}$ or $\|f\|_p$ to denote the Lebesgue L^p norm of f for $1 \leq p \leq \infty$. We use $L^q_t L^r_x$ to denote the space-time norm

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r\ dx\right)^{q/r}\ dt\right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of space-time such as $I\times\mathbb{R}^d$. When $q=r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$ or L_{tx}^q. We shall write $u\in L_{t,loc}^q L_x^r(\mathbb{R}\times\mathbb{R}^d)$ if

$$\|u\|_{L_t^q L_x^r(K\times\mathbb{R}^d)}<\infty, \qquad \text{for any compact } K\subset\mathbb{R}. \tag{2.1}$$

We shall need the standard dispersive inequality: for any $2\le p\le\infty$,

$$\|e^{it\Delta}f\|_p \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})}\|f\|_{\frac{p}{p-1}}, \qquad \forall\, t\neq 0.$$

The dispersive inequality can be used to deduce certain space-time estimates known as Strichartz inequalities. Recall that for dimension $d\ge 1$, we say a pair of exponents (q,r) is *(Schrödinger) admissible* if

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2}, \qquad 2\le q,r\le\infty, \text{ and } (d,q,r)\neq(2,2,\infty).$$

For any fixed space-time slab $I\times\mathbb{R}^d$, we define the Strichartz norm

$$\|u\|_{S(I)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)}. \tag{2.2}$$

For $d=2$, we need to further impose $q>q_1$ in the above norm for some q_1 slightly larger than 2, so as to stay away from the forbidden endpoint. The choice of q_1 is usually simple. We use $S(I)$ to denote the closure of all test functions in $\mathbb{R}\times\mathbb{R}^d$ under this norm. We denote by $N(I)$ the dual space of $S(I)$.

We now state the standard Strichartz estimates. For the non-endpoint case, one can see for example [12]. For the end-point case, see [14].

Lemma 2.1. *If $u:\ I\times\mathbb{R}^d\to\mathbb{C}$ solves*

$$i\partial_t u+\Delta u=F, \quad u(t_0)=u_0,$$

for some $t_0\in I$, $u_0\in L_x^2(\mathbb{R}^d)$. Then

$$\|u\|_{S(I)}\lesssim_d \|u_0\|_2+\|F\|_{N(I)}.$$

We need a few simple estimates on the nonlinearity. For any complex-valued function $F=F(z)$, recall the notation

$$F_z := \frac{1}{2}\left(\frac{\partial F}{\partial x}-i\frac{\partial F}{\partial y}\right), \qquad F_{\bar z} := \frac{1}{2}\left(\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}\right).$$

If we write $F(z)=F^*(z,\bar z)$ with z and $\bar z$ treated as independent variables in F^*, then $F_z=\frac{\partial F^*}{\partial z}$ and $F_{\bar z}=\frac{\partial F^*}{\partial \bar z}$.

By the chain rule and Fundamental Theorem of Calculus, it is easy to check that

$$\begin{aligned}\nabla(F(u(x))) &= F_z(u(x))\nabla u(x)+F_{\bar z}(u(x))\overline{\nabla u(x)};\\ F(z_1)-F(z_2) &= (z_1-z_2)\int_0^1 F_z(z_2+\theta(z_1-z_2))d\theta\\ &\quad+(\overline{z_1-z_2})\int_0^1 F_{\bar z}(z_2+\theta(z_1-z_2))d\theta.\end{aligned} \tag{2.3}$$

These two identities will be used later.

Lemma 2.2 (Hölder continuity of f' and g)**.** *Let $f(z) = g(|z|^2)z$ for $z \in \mathbb{C}$ and suppose g satisfy* (1.9) *and* (1.12)*. Then for all s_1, $s_2 > 0$ we have*

$$\begin{aligned} |g(s_1^2) - g(s_2^2)| + |s_1^2 g'(s_1^2) - s_2^2 g'(s_2^2)| \\ \lesssim |s_1 - s_2|^{\min\{2\alpha_1, 1\}}(s_1 + s_2)^{\max\{2\alpha_1 - 1, 0\}} \\ + |s_1 - s_2|^{\min\{2\alpha_2, 1\}}(s_1 + s_2)^{\max\{2\alpha_2 - 1, 0\}}; \quad (2.4) \end{aligned}$$

and for any z_1, $z_2 \in \mathbb{C}$,

$$\begin{aligned} |f_z(z_1) - f_z(z_2)| + |f_{\bar{z}}(z_1) - f_{\bar{z}}(z_2)| + |g(|z_1|^2) - g(|z_2|^2)| \\ \lesssim |z_1 - z_2|^{\min\{2\alpha_1, 1\}}(|z_1| + |z_2|)^{\max\{2\alpha_1 - 1, 0\}} \\ + |z_1 - z_2|^{\min\{2\alpha_2, 1\}}(|z_1| + |z_2|)^{\max\{2\alpha_2 - 1, 0\}}; \quad (2.5) \end{aligned}$$

$$|f(z_1) - f(z_2)| \lesssim |z_1 - z_2| \cdot \big((|z_1| + |z_2|)^{2\alpha_1} + (|z_1| + |z_2|)^{2\alpha_2}\big). \quad (2.6)$$

Proof of Lemma 2.2. By (1.9), we get for any $s > 0$,

$$|(s^2 g'(s^2))'| \lesssim |s g'(s^2)| + |s^3 g''(s^2)| \lesssim s^{2\alpha_1 - 1} + s^{2\alpha_2 - 1}.$$

Clearly for any s_1, $s_2 > 0$, using the above estimate, we have

$$\begin{aligned} |s_1^2 g'(s_1^2) - s_2^2 g'(s_2^2)| &\lesssim |s_1^{2\alpha_1} - s_2^{2\alpha_1}| + |s_1^{2\alpha_2} - s_2^{2\alpha_2}| \\ &\lesssim \textstyle\sum_{k=1}^2 |s_1 - s_2|^{\min\{2\alpha_k, 1\}}(s_1 + s_2)^{\max\{2\alpha_k - 1, 0\}}. \end{aligned}$$

The estimate for $g(s^2)$ is similar. Therefore (2.4) follows. Observe that

$$f_z(z) = g'(|z|^2)|z|^2 + g(|z|^2), \qquad f_{\bar{z}}(z) = g'(|z|^2)z^2.$$

Obviously (2.5) holds for $g(|z|^2)$ and $f_z(z)$ using (2.4). For $f_{\bar{z}}(z)$, the estimate is similar: Let $z_1 = \rho_1 e^{i\theta_1}$, $z_2 = \rho_2 e^{i\theta_2}$, with $|\theta_1 - \theta_2| \le \pi$. One just need to note that

$$|f_{\bar{z}}(z_1) - f_{\bar{z}}(z_2)| = |g'(\rho_1^2)\rho_1^2 e^{i(\theta_1 - \theta_2)} - g'(\rho_2^2)\rho_2^2 e^{i(\theta_2 - \theta_1)}|,$$

and $|z_1 - z_2| \sim |\rho_1 - \rho_2||\cos(\frac{\theta_1 - \theta_2}{2})| + (\rho_1 + \rho_2)|\sin(\frac{\theta_1 - \theta_2}{2})|$. Estimating the real and imaginary parts separately gives the result. Finally (2.6) follows from (2.3) and (2.5). □

With the preliminaries and notations out of the way, we now turn to the main matter of this section.

To prove our results, we shall state and prove a general proposition on the solvability of NLS around an approximate solution profile with exponentially decaying source terms. This proposition is very useful in that it reduces the construction of multi-soliton solutions to the verification of only a few conditions (see (2.7) and (2.11) below). To simplify numerology we shall first deal with the pure power nonlinearity case.

Proposition 2.3. *Let $0 < \alpha < \alpha_{\max}$. Let $H = H(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$, $W = W(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ be given functions which satisfy for some $C_1 > 0$, $\lambda > 0$:*

$$\|W(t)\|_{\alpha+2} + e^{\lambda t}\|H(t)\|_{\frac{\alpha+2}{\alpha+1}} \le C_1, \qquad \forall\, t \ge 0. \quad (2.7)$$

Let $f_1(z) = |z|^\alpha z$ and consider the equation

$$\eta(t) = i\int_t^\infty e^{i(t-\tau)\Delta}\Big(f_1(W + \eta) - f_1(W) + H\Big)(\tau)\, d\tau. \quad (2.8)$$

There exists a constant $\lambda_* = \lambda_*(\alpha, d, C_1) > 0$ sufficiently large such that if $\lambda \ge \lambda_*$ then the following holds:

- *There exists a unique solution η to (2.8) satisfying*

$$\|\eta(t)\|_{\alpha+2} \le C_1 e^{-\lambda t}, \qquad \forall\, t \ge 0. \tag{2.9}$$

- *All (L^2 level) Strichartz norms of η are finite and decay exponentially, i.e.*

$$\|\eta\|_{S([t,\infty))} \lesssim e^{-\lambda t}, \qquad \forall\, t \ge 0. \tag{2.10}$$

- *If in addition to (2.7), (H, W) also satisfies for some $C_2 > 0$:*

$$\|\nabla W(t)\|_{\alpha+2} + e^{\lambda t}\|\nabla H(t)\|_{\frac{\alpha+2}{\alpha+1}} \le C_2, \quad \forall\, t \ge 0, \tag{2.11}$$

then $\eta \in L_t^\infty H_x^1$, and for some $C_3 = C_3(d, \alpha, C_1) > 0$,

$$\|\nabla \eta(t)\|_{\alpha+2} + \|\nabla\eta\|_{S([t,\infty))} \le C_3 C_2 e^{-\min\{\alpha,1\}\lambda t}, \quad \forall\, t \ge 0. \tag{2.12}$$

Here both C_3 and λ_* are independent of C_2.

Proof of Proposition 2.3. We write (2.8) as $\eta = V\eta$. We shall show that for λ sufficiently large, V is a contraction in the ball

$$B = \left\{ \eta : \ \|\eta\|_{\tilde{X}} := \left\| e^{\lambda t} \|\eta(t)\|_{\alpha+2} \right\|_{L_t^\infty([0,\infty))} \le C_1 \right\}.$$

We first check that V maps B into B. Denote

$$\theta := d\left(\frac{1}{2} - \frac{1}{\alpha+2}\right).$$

It is easy to check that $0 < \theta < 1$ since by assumption $0 < \alpha < \alpha_{\max}$. By the simple inequality

$$|f_1(z_1) - f_1(z_2)| \lesssim |z_1 - z_2| \cdot (|z_1|^\alpha + |z_2|^\alpha), \qquad \forall\, z_1,\, z_2 \in \mathbb{C} \tag{2.13}$$

we have

$$|f_1(W+\eta) - f_1(W)| \lesssim |\eta| \cdot (|W|^\alpha + |\eta|^\alpha). \tag{2.14}$$

By using the dispersive estimate, the assumptions on $(W,\, H)$ and (2.14), we have

$$\begin{aligned}
&\|\eta(t)\|_{\alpha+2} \\
&\le C \int_t^\infty |t-\tau|^{-\theta} \Big(\||W(\tau)|^\alpha |\eta(\tau)|\|_{\frac{\alpha+2}{\alpha+1}} + \||\eta(\tau)|^{\alpha+1}\|_{\frac{\alpha+2}{\alpha+1}} + \|H(\tau)\|_{\frac{\alpha+2}{\alpha+1}} \Big) d\tau \\
&\le C \int_t^\infty |t-\tau|^{-\theta} \Big(\|W(\tau)\|_{\alpha+2}^\alpha \|\eta(\tau)\|_{\alpha+2} + \|\eta(\tau)\|_{\alpha+2}^{\alpha+1} + \|H(\tau)\|_{\frac{\alpha+2}{\alpha+1}} \Big) d\tau \\
&\le C \int_t^\infty |t-\tau|^{-\theta} \Big(C_1^\alpha C_1 e^{-\lambda\tau} + C_1^{\alpha+1} e^{-\lambda(\alpha+1)\tau} + C_1 e^{-\lambda\tau} \Big) d\tau \\
&\le C C_1 e^{-\lambda t} I_1,
\end{aligned} \tag{2.15}$$

where $C = C(d, \alpha)$ and ($\tilde{\tau} = \tau - t$)

$$I_1 = C_1^\alpha \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda\tilde{\tau}} d\tilde{\tau} + C_1^\alpha \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda(\alpha+1)\tilde{\tau}} d\tilde{\tau} + \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda\tilde{\tau}} d\tilde{\tau}.$$

It is not difficult to check that for λ sufficiently large

$$CI_1 \le \frac{C(C_1, d, \alpha)}{\lambda^{1-\theta}} \le 1.$$

Hence $\|\eta(t)\|_{\alpha+2} \le C_1 e^{-\lambda t}$ and V maps B to B. By using (2.13) and a similar estimate as in (2.15), we can also show that for any $\eta_1 \in B$, $\eta_2 \in B$,

$$\|(V\eta_1)(t) - (V\eta_2)(t)\|_{\tilde{X}} \le \frac{1}{2}\|\eta_1 - \eta_2\|_{\tilde{X}}.$$

This completes the proof that V is a contraction on B.

Next (2.10) is a simple consequence of the Strichartz estimate. Denote by a the number such that $\frac{2}{a} + \frac{d}{\alpha+2} = \frac{d}{2}$. It is easy to check that $2 < a < \infty$ since $0 < \alpha < \alpha_{\max}$. By (2.13) and Strichartz estimate, we have

$$\begin{aligned}
\|\eta\|_{S([t,\infty))} &\lesssim \|f_1(W+\eta) - f_1(W)\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} + \|H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} \\
&\lesssim \||\eta| \cdot (|W|^\alpha + |\eta|^\alpha)\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} + \|H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} \\
&\lesssim \|W\|^\alpha_{L_\tau^\infty L_x^{\alpha+2}([0,\infty))} \cdot \|\eta\|_{L_\tau^{\frac{a}{a-1}} L_x^{\alpha+2}([t,\infty))} \\
&\quad + \|\eta\|^{\alpha+1}_{L_\tau^{\frac{(\alpha+1)a}{a-1}} L_x^{\alpha+2}([t,\infty))} + \|H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} \\
&\lesssim e^{-\lambda t}, \qquad \forall\, t \ge 0.
\end{aligned} \tag{2.16}$$

Finally to show (2.12), we first prove that V maps B_1 into B_1 where

$$B_1 = B \bigcap \left\{ \eta : \sup_{t \ge 0} \Big(e^{\min\{\alpha,1\}\lambda t} \|\nabla \eta(t)\|_{\alpha+2} \Big) \le C_2 \right\}.$$

We start with the identity

$$\begin{aligned}
&\nabla\big(f_1(W+\eta) - f_1(W)\big) \\
&= \big((\partial_z f_1)(W+\eta) - (\partial_z f_1)(W)\big)\nabla(W+\eta) + (\partial_z f_1)(W)\nabla\eta \\
&\quad + \big((\partial_{\bar z} f_1)(W+\eta) - (\partial_{\bar z} f_1)(W)\big)\overline{\nabla(W+\eta)} + (\partial_{\bar z} f_1)(W)\overline{\nabla\eta}.
\end{aligned} \tag{2.17}$$

Note that for $0 < \alpha \le 1$,

$$|(\partial_z f_1)(z_1) - (\partial_z f_1)(z_2)| \lesssim |z_1 - z_2|^\alpha, \qquad \forall\, z_1,\, z_2 \in \mathbb{C},$$

and for $\alpha > 1$,

$$|(\partial_z f_1)(z_1) - (\partial_z f_1)(z_2)| \lesssim (|z_1|^{\alpha-1} + |z_2|^{\alpha-1})|z_1 - z_2|, \qquad \forall\, z_1,\, z_2 \in \mathbb{C}.$$

Therefore

$$|\nabla(f_1(W+\eta) - f_1(W))| \lesssim \begin{cases} |\eta|^\alpha |\nabla(W+\eta)| + |W|^\alpha |\nabla\eta|, & \text{if } 0 < \alpha \le 1, \\ (|\eta|^{\alpha-1} + |W|^{\alpha-1})|\eta||\nabla(W+\eta)| + |W|^\alpha|\nabla\eta|, & \text{if } \alpha > 1 \end{cases} \tag{2.18}$$

For simplicity we shall only discuss the case $0 < \alpha \le 1$. The argument for $\alpha > 1$ is similar (even simpler) and will be omitted. By using (2.18) , (2.9), (2.11), and the dispersive inequality, we have for $t \ge 0$:

$$\begin{aligned}
\|\nabla \eta(t)\|_{\alpha+2} &\lesssim_{d,\alpha} \int_t^\infty |t-\tau|^{-\theta} \Big(\||\eta|^\alpha |\nabla(W+\eta)|\|_{\frac{\alpha+2}{\alpha+1}} + \||W|^\alpha \nabla \eta\|_{\frac{\alpha+2}{\alpha+1}} + \|\nabla H\|_{\frac{\alpha+2}{\alpha+1}} \Big) d\tau \\
&\lesssim_{d,\alpha} \int_t^\infty |t-\tau|^{-\theta} \Big(\|\eta\|_{\alpha+2}^\alpha (\|\nabla W\|_{\alpha+2} + \|\nabla \eta\|_{\alpha+2}) \\
&\qquad\qquad + \|W\|_{\alpha+2}^\alpha \|\nabla \eta\|_{\alpha+2} + \|\nabla H\|_{\frac{\alpha+2}{\alpha+1}} \Big) d\tau \\
&\lesssim_{d,\alpha,C_1} C_2 \int_t^\infty |t-\tau|^{-\theta} e^{-\lambda\alpha\tau} d\tau + C_2 \int_t^\infty |t-\tau|^{-\theta} e^{-\lambda\tau} d\tau \\
&\lesssim_{d,\alpha,C_1} C_2 \int_t^\infty |t-\tau|^{-\theta} e^{-\lambda\alpha\tau} d\tau \\
&\le C_2 e^{-\lambda\alpha t} \cdot C(d,\alpha,C_1) \int_0^\infty |\tilde{\tau}|^{-\theta} e^{-\lambda\alpha\tilde{\tau}} d\tilde{\tau} \\
&= C_2 e^{-\lambda\alpha t} \cdot C(d,\alpha,C_1) \cdot (\lambda\alpha)^{-(1-\theta)} \int_0^\infty |\tilde{\tau}|^{-\theta} e^{-\tilde{\tau}} d\tilde{\tau}.
\end{aligned}$$

Now if we take $\lambda \ge \lambda_*$ and $\lambda_* = \lambda_*(d,\alpha,C_1)$ is independent of C_2 and sufficiently large such that

$$C(d,\alpha,C_1) \cdot (\lambda_*\alpha)^{-(1-\theta)} \int_0^\infty |\tilde{\tau}|^{-\theta} e^{-\tilde{\tau}} d\tilde{\tau} \le \frac{1}{2}, \tag{2.19}$$

then clearly

$$\|\nabla \eta(t)\|_{\alpha+2} \le C_2 e^{-\lambda\alpha t}, \qquad \forall\, t \ge 0.$$

By a similar argument, we also obtain for the case $\alpha > 1$,

$$\|\nabla \eta(t)\|_{\alpha+2} \le C_2 e^{-\lambda t}, \qquad \forall\, t \ge 0.$$

Hence we have proved that V maps B_1 to B_1. Since V is a contraction on B and maps B_1 into B_1, it is obvious that we have constructed the solution satisfying

$$\|\nabla \eta(t)\|_{\alpha+2} \le C_2 e^{-\lambda \min\{\alpha,1\} t}, \qquad \forall\, t \ge 0. \tag{2.20}$$

It remains for us to bound the Strichartz norm $\|\nabla \eta(t)\|_{S([t,\infty))}$. The argument is similar to that in (2.16). Let a be the same number such that $\frac{2}{a} + \frac{d}{\alpha+2} = \frac{d}{2}$. By (2.18) and Strichartz, we have

$$\begin{aligned}
\|\nabla \eta\|_{S([t,\infty))} &\lesssim_d \Big\| |\eta|^\alpha |\nabla(W+\eta)| \Big\|_{N([t,\infty))} + \Big\| |W|^\alpha |\nabla \eta| \Big\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))} \\
&\lesssim_d \Big\| |\eta|^\alpha |\nabla W| \Big\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} + \Big\| |\eta|^\alpha |\nabla \eta| \Big\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} \\
&\quad + \||W|^\alpha |\nabla \eta|\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} + \|\nabla H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))} \\
&\lesssim_d \Big\| |\eta|^\alpha \Big\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha}}([t,\infty))} \Big\| |\nabla W| + |\nabla \eta| \Big\|_{L_\tau^\infty L_x^{\alpha+2}([t,\infty))} \\
&\qquad + \||W|^\alpha\|_{L_\tau^\infty L_x^{\frac{\alpha+2}{\alpha}}([t,\infty))} \|\nabla \eta\|_{L_\tau^{\frac{a}{a-1}} L_x^{\alpha+2}([t,\infty))} \\
&\qquad\qquad + \|\nabla H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha+1}}([t,\infty))}.
\end{aligned} \tag{2.21}$$

By (2.9), we have

$$\begin{aligned}
\Big\| |\eta|^\alpha \Big\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{\alpha+2}{\alpha}}([t,\infty))} &\le \Big\| \|\eta\|_{\alpha+2}^\alpha \Big\|_{L_\tau^{\frac{a}{a-1}}([t,\infty))} \\
&\le C_1^\alpha \Big(\int_t^\infty e^{-\lambda\alpha \frac{a\tau}{a-1}} d\tau \Big)^{\frac{a-1}{a}} \\
&\le C_1^\alpha \cdot \left(\lambda \alpha \frac{a}{a-1} \right)^{-\frac{a-1}{a}} \cdot e^{-\lambda\alpha t}.
\end{aligned}$$

Plugging the above estimates into (2.21) and using (2.11), (2.20), we obtain

$$\|\nabla \eta\|_{S([t,\infty))} \lesssim_{d,\alpha,C_1} C_2 e^{-\lambda\alpha t}.$$

This settles the estimate for $0 < \alpha \le 1$.

By a similar estimate, we also have for $\alpha > 1$,

$$\|\nabla \eta\|_{S([t,\infty))} \lesssim_{d,\alpha,C_1} C_2 e^{-\lambda t}.$$

This completes the proof of (2.12). □

The next proposition, unlike Proposition 2.3, is based solely on Strichartz estimates. It will be used in the proof of Theorems 1.9 and 1.16. Several assumptions and conditions have to be modified to take care of the general nonlinearity $f(u)$.

Proposition 2.4. *Let f be the same as in* (1.1) *satisfying condition* (1.9)*. Let $H = H(t,x)$: $[0,\infty) \times \mathbb{R}^d \to \mathbb{C}$, $W = W(t,x)$: $[0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ be given functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \ge 0$:*

$$\begin{aligned}
&\|W(t)\|_\infty + e^{\lambda t}\|H(t)\|_2 \le C_1, \qquad \forall\, t \ge T_0; \\
&\|\nabla W(t)\|_2 + \|\nabla W(t)\|_\infty + e^{\lambda t}\|\nabla H(t)\|_2 \le C_2, \qquad \forall\, t \ge T_0.
\end{aligned} \tag{2.22}$$

Consider the equation

$$\eta(t) = i \int_t^\infty e^{i(t-\tau)\Delta} \Big(f(W+\eta) - f(W) + H \Big)(\tau)\, d\tau, \qquad t \ge T_0. \tag{2.23}$$

There exists a constant $\lambda_* = \lambda_*(d, \alpha_1, \alpha_2, C_1) > 0$ (independent of C_2) and a time $T_* = T_*(d, \alpha_1, \alpha_2, C_1, C_2) > 0$ sufficiently large such that if $\lambda \ge \lambda_*$ and $T_0 \ge T^*$, then there exists a unique solution η to (2.23) *on $[T_0, +\infty) \times \mathbb{R}^d$ satisfying*

$$e^{\lambda t}\|\eta\|_{S([t,\infty))} + e^{\lambda c_1 t}\|\nabla \eta\|_{S([t,\infty))} \le 1, \qquad \forall t \ge T_0. \tag{2.24}$$

Here $c_1 > 0$ is a constant depending only on (α_1, d).

Remark 2.5. It is important to notice that λ_* *does not depend* on C_2. This will be essential for the proof of Theorems 1.9 and 1.16.

Proof of Proposition 2.4. To minimize numerology we will suppress all explicit dependence of constants on all parameters except the constant C_2.

We now sketch the main computations. Take $0 < \beta_1 \le 2\alpha_1$ such that $\beta_1 < \frac{1}{100d}$. Denote

$$\beta_2 := \begin{cases} \frac{4}{d-2}, & \text{if } d \ge 3, \\ m-1, & \text{if } d = 1, 2; \end{cases}$$
$$c_1 := \frac{1}{2}\beta_1.$$

Here for $d = 1, 2$, m is an integer such that $m > 2\alpha_2 + 2$.

We shall omit the standard contraction argument since it will be essentially a repetition and we check only the following property: If on $[T_0, +\infty)$ we have

$$e^{\lambda t}\|\eta\|_{S([t,\infty))} + e^{c_1\lambda t}\|\nabla\eta\|_{S([t,\infty))} \le C.$$

then the following a priori estimate holds, provided λ and T_0 are chosen large enough,

$$e^{\lambda t}\|\eta\|_{S([t,\infty))} + e^{c_1\lambda t}\|\nabla\eta\|_{S([t,\infty))} \le 1. \tag{2.25}$$

We start with $\|\eta\|_{S([t,\infty))}$. By Lemma 2.2 and Strichartz, we have

$$\begin{aligned} \|\eta\|_{S([t,\infty))} &\lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))} + \|H\|_{N([t,\infty))} \\ &\lesssim \|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})\|_{N([t,\infty))} \qquad (2.26) \\ &\quad + \|H\|_{L^1_\tau L^2_x([t,\infty))}. \qquad (2.27) \end{aligned}$$

For (2.27), by using (2.22), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau} d\tau \le \frac{1}{100} e^{-\lambda t},$$

where the constant $\frac{1}{100}$ is obtained by taking λ large enough.
For (2.26), consider two cases. If $d \ge 3$, then by the boundedness of W, we have

$$\left|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})\right| \lesssim |\eta| + |\eta|^{1+\frac{4}{d-2}}. \tag{2.28}$$

Hence for $d \ge 3$, using that both $(\frac{2d+4}{d}, \frac{2d+4}{d})$ and (q^*, q) are admissible with $1/q* = 1/q - 1/d = \frac{d-2}{2d+4}$,

$$\begin{aligned} (2.26) &\lesssim \|\eta\|_{L^1_\tau L^2_x([t,\infty))} + \|\eta|\eta|^{\frac{4}{d-2}}\|_{L_{\tau,x}^{\frac{2(d+2)}{d+4}}([t,\infty))} \\ &\lesssim \int_t^\infty e^{-\lambda\tau} d\tau + \|\eta\|_{L_{\tau,x}^{\frac{2(d+2)}{d}}([t,\infty))} \cdot \|\eta\|^{\frac{4}{d-2}}_{L_{\tau,x}^{\frac{2(d+2)}{d-2}}([t,\infty))} \\ &\lesssim \frac{1}{\lambda} e^{-\lambda t} + \|\eta\|_{S([t,\infty))} \cdot \|\nabla\eta\|^{\frac{4}{d-2}}_{S([t,\infty))} \\ &\lesssim \frac{1}{\lambda} e^{-\lambda t} + e^{-\lambda t} \cdot e^{-\frac{4}{d-2}c_1\lambda t} \\ &\le \frac{1}{100} e^{-\lambda t}, \end{aligned}$$

where we have used the fact that λ and $t \ge T_0$ are sufficiently large.
For $d = 1, 2$, we replace (2.28) by

$$|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})| \lesssim |\eta| + |\eta|^m.$$

Then

$$\||\eta|^m\|_{N([t,\infty))} \lesssim \||\eta|^m\|_{L^1_\tau L^2_x([t,\infty))}$$
$$\lesssim \int_t^\infty \|\eta(\tau)\|_{2m}^m d\tau.$$

By (2.24) and interpolation (i.e. Gagliardo-Nirenberg), we have for $\theta = d(\frac{1}{2} - \frac{1}{2m})$

$$\|\eta(\tau)\|_{2m} \lesssim \|\eta(\tau)\|_2^{1-\theta} \|\nabla\eta(\tau)\|_2^\theta$$
$$\lesssim e^{-\left((1-\theta)\lambda + c_1\lambda\theta\right)\tau}.$$

It is easy to check that $m(1-\theta) \geq 1$. Therefore

$$\||\eta|^m\|_{N([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau} d\tau \leq \frac{1}{100} e^{-\lambda t}.$$

Hence the estimate also holds for $d = 1, 2$. Consequently for all $d \geq 1$, and $t \geq T_0$,

$$\|\eta\|_{S([t,\infty))} \leq \frac{1}{10} e^{-\lambda t}.$$

Now we estimate $\|\nabla\eta\|_{S([t,\infty))}$. By Strichartz and (2.17)

$$\begin{aligned}
\|\nabla\eta\|_{S([t,\infty))} &\lesssim \|\nabla(f(W+\eta) - f(W))\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))} \\
&\lesssim \||f_z(W+\eta) - f_z(W)| \cdot \nabla(W+\eta)\|_{N([t,\infty))} \\
&\quad + \||f_{\bar{z}}(W+\eta) - f_{\bar{z}}(W)| \cdot \overline{\nabla(W+\eta)}\|_{N([t,\infty))} \\
&\quad + \||f_z(W)|\nabla\eta\|_{N([t,\infty))} + \||f_{\bar{z}}(W)|\overline{\nabla\eta}\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))}.
\end{aligned}$$

By Lemma 2.2, we get

$$\|\nabla\eta\|_{S([t,\infty))} \lesssim \||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))} + \||\eta|^{\beta_1}|\nabla W|)\|_{N([t,\infty))} \tag{2.29}$$
$$+ \||\eta|^{\min\{\beta_2,1\}}(|W| + |\eta|)^{\max\{\beta_2-1,0\}} \cdot (|\nabla W| + |\nabla\eta|)\|_{N([t,\infty))} \tag{2.30}$$
$$+ \|(|f_z(W)| + |f_{\bar{z}}(W)|)\nabla\eta\|_{L^1_\tau L^2_x([t,\infty))} + \|\nabla H\|_{L^1_\tau L^2_x([t,\infty))}. \tag{2.31}$$

Consider (2.29). Let a be the number such that $\frac{2}{a} + \frac{d}{\beta_1+2} = \frac{d}{2}$ and let $a' = \frac{a}{a-1}$. Then

$$\begin{aligned}
\||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))} &\lesssim \||\eta|^{\beta_1}\nabla\eta\|_{L^{a'}_\tau L_x^{\frac{\beta_1+2}{\beta_1+1}}([t,\infty))} \\
&\lesssim \||\eta|^{\beta_1}\|_{L_\tau^{(\frac{1}{a'}-\frac{1}{a})^{-1}} L_x^{\frac{\beta_1+2}{\beta_1}}([t,\infty))} \|\nabla\eta\|_{L^a_\tau L_x^{\beta_1+2}([t,\infty))} \\
&\lesssim \left(\int_t^\infty \|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot\frac{a}{a-2}} d\tau\right)^{\frac{a-2}{a}} \cdot \|\nabla\eta\|_{S([t,\infty))}.
\end{aligned} \tag{2.32}$$

It is not difficult to check that $\beta_1 \cdot \frac{a}{a-2} < a$ (since $\beta_1 < 4/d$). By using the fact $\|\eta\|_{L^a_\tau L_x^{\beta_1+2}([t,\infty))} \lesssim e^{-\lambda t}$ and Hölder inequality, for $t \geq T_0$ we have

$$\begin{aligned}
\int_t^\infty \|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot\frac{a}{a-2}} d\tau &\lesssim \sum_{k\geq t-1} \int_k^{k+1} \|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot\frac{a}{a-2}} d\tau \\
&\lesssim \sum_{k\geq t-1} \left(\int_k^{k+1} \|\eta(\tau)\|_{\beta_1+2}^a d\tau\right)^{\frac{1}{a}\cdot\frac{a\beta_1}{a-2}} \\
&\lesssim \sum_{k\geq t-1} e^{-\lambda k\cdot\frac{a\beta_1}{a-2}} \lesssim \frac{1}{\lambda} e^{-\lambda(t-1)\cdot\frac{a\beta_1}{a-2}}.
\end{aligned}$$

Plugging the above estimate into (2.32), we obtain

$$\||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))} \lesssim \left(\frac{1}{\lambda}\right)^{\frac{a-2}{a}} e^{-\lambda\beta_1(t-1)}\cdot e^{-c_1\lambda t} \le \frac{1}{100}e^{-c_1\lambda t}, \qquad t\ge T_0,$$

for λ sufficiently large and $T_0\ge 1$.
Similarly we have for $t\ge T_0$, using $\beta_1 a' = \beta_1 a/(a-1) < a$,

$$\begin{aligned}
\||\eta|^{\beta_1}|\nabla W|\|_{N([t,\infty))} &\lesssim \||\eta|^{\beta_1}\|_{L_\tau^{a'}L_x^{\frac{\beta_1+2}{\beta_1}}([t,\infty))}\|\nabla W\|_{L_\tau^\infty L_x^{\beta_1+2}([t,\infty))}\\
&\lesssim e^{-\lambda\beta_1(t-1)}C_2\\
&\lesssim e^{-c_1\lambda t}e^{-\lambda c_1(t-2)}C_2 \le \frac{1}{100}e^{-c_1\lambda t}.
\end{aligned}$$

Hence

$$(2.29)\le \frac{1}{50}e^{-c_1\lambda t}.$$

Next we deal with (2.30). Consider first the case $d\ge 6$. In this case $\beta_2\le 1$. Therefore

$$\begin{aligned}
(2.30) &\lesssim \||\eta|^{\frac{4}{d-2}}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\eta|^{\frac{4}{d-2}}\nabla\eta\|_{L_{\tau,x}^{\frac{2(d+2)}{d+4}}([t,\infty))} + \||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{L_\tau^2 L_x^{\frac{2d}{d+2}}([t,\infty))}\\
&\lesssim \|\nabla\eta\|_{S([t,\infty))}^{1+\frac{4}{d-2}} + \left\|\|\eta(\tau)\|_{L_x^2}^{\frac{4}{d-2}}\cdot\|\nabla W\|_{L_x^{(\frac{d+2}{2d}-\frac{2}{d-2})^{-1}}}\right\|_{L_\tau^2([t,\infty))}\\
&\lesssim e^{-c_1\lambda(1+\frac{4}{d-2})t} + C_2\cdot\left(\int_t^\infty \|\eta(\tau)\|_2^{\frac{4}{d-2}\cdot 2}d\tau\right)^{\frac12}\\
&\lesssim e^{-c_1\lambda(1+\frac{4}{d-2})t} + C_2\cdot\left(\int_t^\infty e^{-\frac{8}{d-2}\lambda\tau}d\tau\right)^{\frac12}\\
&\le \frac{1}{200}e^{-c_1\lambda t} + C_2\cdot e^{-c_1\lambda T_0}\cdot e^{-c_1\lambda t}\le \frac{1}{100}e^{-c_1\lambda t},
\end{aligned} \tag{2.33}$$

for λ and T_0 sufficiently large.
Consider next the case $3\le d\le 5$. In this case $\beta_2 = \frac{4}{d-2} > 1$. Therefore using the boundedness of W, we have

$$\begin{aligned}
(2.30) &\lesssim \||\eta|\cdot(|W|+\eta)^{\frac{4}{d-2}-1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\eta|^{\beta_1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))} + \||\eta|^{\frac{4}{d-2}}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim |(2.29)| + \||\eta|^{\frac{4}{d-2}}\nabla\eta\|_{L_{\tau,x}^{\frac{2(d+2)}{d+4}}([t,\infty))} + \||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{N([t,\infty))}\\
&\le \frac{1}{30}e^{-c_1\lambda t} + \||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{N([t,\infty))}.
\end{aligned}$$

For $d=5$, we can bound the term $\||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{N([t,\infty))}$ in the same way as in (2.33) (it is easy to check that $\frac{2d}{d+2}<\frac{d-2}{2}$ for $d\ge 5$). For $d=3,4$, we have

$$\begin{aligned}
\||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{N([t,\infty))} &\lesssim \||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{L_\tau^2 L_x^{\frac{2d}{d+2}}([t,\infty))}\\
&\lesssim C_2\Big(\int_t^\infty \|\eta(\tau)\|_{\frac{8d}{d^2-4}}^{\frac{8}{d-2}}d\tau\Big)^{\frac12}.
\end{aligned} \tag{2.34}$$

Since $d = 3,4$, it is easy to check that $2 < \frac{8d}{d^2-4} < \frac{2d}{d-2}$. By interpolation we have for $\theta = \frac{1}{8}(d-2)^2$,

$$\begin{aligned}\|\eta(\tau)\|_{L_x^{\frac{8d}{d^2-4}}} &\lesssim \|\eta(\tau)\|_2^{\theta}\|\nabla\eta(\tau)\|_2^{1-\theta}\\ &\lesssim e^{-\theta\lambda\tau}e^{-(1-\theta)c_1\lambda\tau} \lesssim e^{-\theta\lambda\tau}.\end{aligned}$$

Plugging this estimate into (2.34), we obtain for $d = 3,4$,

$$\||\eta|^{\frac{4}{d-2}}|\nabla W|\|_{N([t,\infty))} \lesssim C_2\Big(\int_t^\infty e^{-\lambda(d-2)\tau}\,d\tau\Big)^{\frac{1}{2}} \lesssim C_2\cdot\lambda^{-\frac{d-2}{2}}e^{-\frac{d-2}{2}\lambda t} \le \frac{1}{100}e^{-c_1\lambda t}$$

which is clearly enough for us.
It remains to bound (2.30) for $d = 1,2$. Since in this case $\beta_2 = m-1 > 1$, we have

$$\begin{aligned}(2.30) &\lesssim \||\eta|(|W|+|\eta|)^{m-2}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\ &\lesssim \||\eta|^{\beta_1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))} + \||\eta|^m|\nabla W|\|_{N([t,\infty))} + \||\eta|^m|\nabla\eta|\|_{N([t,\infty))}\\ &\lesssim |(2.29)| + \||\eta|^m|\nabla W|\|_{L_\tau^1L_x^2([t,\infty))} + \||\eta|^m|\nabla\eta|\|_{L_{\tau,x}^{\frac{2(d+2)}{d+4}}([t,\infty))}\\ &\lesssim |(2.29)| + C_2\|\eta\|^m_{L_\tau^mL_x^{2m}([t,\infty))} + \|\nabla\eta\|_{S([t,\infty))}\cdot\|\eta\|^m_{L_{\tau,x}^{m\cdot\frac{d+2}{2}}([t,\infty))}.\end{aligned} \tag{2.35}$$

Now by Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\eta(\tau)\|_{2m}^m &\lesssim \Big(\|\eta(\tau)\|_2^{1-d(\frac{1}{2}-\frac{1}{2m})}\|\nabla\eta(\tau)\|_2^{d(\frac{1}{2}-\frac{1}{2m})}\Big)^m\\ &\lesssim \|\eta(\tau)\|_2^{\frac{d}{2}} \lesssim e^{-\frac{1}{2}\lambda\tau}.\end{aligned}$$

Similarly

$$\|\eta(\tau)\|^m_{\frac{m(d+2)}{2}} \lesssim \|\eta(\tau)\|_2^{\frac{2d}{d+2}} \lesssim e^{-\frac{1}{2}\lambda\tau}.$$

Plugging the above estimates into (2.35) and integrating in time, we obtain for $d = 1,2$,

$$(2.30) \le \frac{1}{100}e^{-c_1\lambda t}$$

which is acceptable for us. We have completed the estimate of (2.30) for all $d \ge 1$.
Finally consider (2.31). Note $\||f_z(W)| + |f_{\bar z}(W)|\|_{L^\infty_{t,x}} \le C$ by (2.5) and (2.22). Thus

$$\begin{aligned}(2.31) &\le C\int_t^\infty \Big(\|\nabla\eta\|_{L_t^\infty L_x^2([\tau,\infty))} + \|\nabla H(\tau)\|_{L_x^2}\Big)\,d\tau\\ &\le C\int_t^\infty (e^{-c_1\lambda\tau} + C_2e^{-\lambda\tau})\,d\tau \le \left(\frac{C}{c_1\lambda} + C_2e^{-c_1\lambda t}\right)e^{-c_1\lambda t} \le \frac{1}{100}e^{-c_1\lambda t}\end{aligned}$$

if we take λ and $t \ge T_0$ large enough.
We have finished the proof of the a priori estimate (2.25). The proposition is proved. $\square$

Remark 2.6. Our proof does not work for the energy-critical case because the overlap of multi-solitons no longer decays exponentially, but is just power-like; our proof relies heavily on the exponential decay property.

3 The N-soliton case

In this section we give the proofs of Theorem 1.7 and Theorem 1.9.

We first recall (1.14), the multi-soliton profile, and observe that the difference $\eta = u - R$ satisfies the equation

$$
\begin{aligned}
i\partial_t \eta + \Delta \eta &= -f(R+\eta) + \sum_{j=1}^N f(R_j) \\
&= -\big(f(R+\eta) - f(R)\big) - \big(f(R) - \sum_{j=1}^N f(R_j)\big).
\end{aligned} \tag{3.1}
$$

The following lemma gives the estimates on R and the source term $f(R) - \sum_{j=1}^N f(R_j)$.

Lemma 3.1. *There exist constants $\tilde{C}_1 > 0$ depending on $\big(N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N\big)$, $\tilde{c}_1 > 0$ depending only on α_1, $\tilde{C}_2 > 0$ depending on $\big(N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (v_j)_{j=1}^N, (x_j)_{j=1}^N\big)$, such that the following hold: For every $1 \le r \le \infty$ and $t \ge 0$,*

$$
\|R(t)\|_r + \sum_{j=1}^N \|R_j(t)\|_r \le \tilde{C}_1, \tag{3.2}
$$

$$
\Big\| f(R(t)) - \sum_{j=1}^N f(R_j(t)) \Big\|_r \le \tilde{C}_1 e^{-\tilde{c}_1 \sqrt{\omega_\star} v_\star t}, \tag{3.3}
$$

$$
\|\nabla R(t)\|_r \le \tilde{C}_2, \tag{3.4}
$$

$$
\Big\| \nabla\big(f(R(t)) - \sum_{j=1}^N f(R_j(t))\big) \Big\|_r \le \tilde{C}_2 e^{-\tilde{c}_1 \sqrt{\omega_\star} v_\star t}. \tag{3.5}
$$

Here recall $\omega_\star = \min\{\omega_j,\ 1 \le j \le N\}$ and $v_\star = \min\{|v_k - v_j| :\ 1 \le k \ne j \le N\}$.

Proof of Lemma 3.1. The estimates (3.2) and (3.4) follow directly from (1.10) and (1.13).

To simplify the notations, denote

$$
\Omega := \big(N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N\big).
$$

To prove (3.3), we start with the point-wise estimate. By (3.2) and Lemma 2.2,

$$
\begin{aligned}
\Big| f(R(t,x)) - \sum_{j=1}^N f(R_j(t,x)) \Big| &= \Big| \sum_{j=1}^N g(|R(t,x)|^2) R_j(t,x) - \sum_{j=1}^N g(|R_j(t,x)|^2) R_j(t,x) \Big| \\
&\le \sum_{j=1}^N |g(|R(t,x)|^2) - g(|R_j(t,x)|^2)| \cdot |R_j(t,x)| \\
&\lesssim_\Omega \sum_{j=1}^N \big(|R(t,x) - R_j(t,x)| + |R(t,x) - R_j(t,x)|^{2\alpha_1}\big) \cdot |R_j(t,x)| \\
&\lesssim_\Omega \sup_{k \ne j} \Big(|R_k(t,x)| \cdot |R_j(t,x)| + (|R_k(t,x)| \cdot |R_j(t,x)|)^{2\alpha_1} \Big).
\end{aligned} \tag{3.6}
$$

It suffices to treat the first term in the bracket of (3.6). The second term is similarly estimated.

By (1.13), for any $\delta < 1$,

$$|R_k(t,x)| \lesssim_{d,\delta} e^{-\delta\sqrt{\omega_k}|x-v_kt-x_k|}, \quad \forall\, k = 1, \dots, N.$$

Now fix some $\delta < 1$ for the rest of the proof.

Clearly for any $k \neq j$,

$$|R_k(t,x)| \cdot |R_j(t,x)| \lesssim_{d,\delta} e^{-\delta(\sqrt{\omega_k}|x-v_kt-x_k|+\sqrt{\omega_j}|x-v_jt-x_j|)}. \tag{3.7}$$

By the triangle inequality, it is clear that for all $j \neq k$, $x \in \mathbb{R}^d$, $t \geq 0$:

$$\begin{aligned}
&\sqrt{\omega_k}|x - v_kt - x_k| + \sqrt{\omega_j}|x - v_jt - x_j| \\
\geq &\min\{\sqrt{\omega_j}, \sqrt{\omega_k}\}\Big(|v_j - v_k|t - |x_k - x_j|\Big) \\
\geq &\sqrt{\omega_\star}\Big(v_\star t - |x_k - x_j|\Big).
\end{aligned} \tag{3.8}$$

Plugging (3.8) into (3.7), we obtain for any $k \neq j$,

$$|R_k(t,x)| \cdot |R_j(t,x)| \lesssim_\Omega e^{-\frac{\delta}{2}\sqrt{\omega_\star}v_\star t} \cdot e^{-\frac{\delta}{2}(\sqrt{\omega_k}|x-v_kt-x_k|+\sqrt{\omega_j}|x-v_jt-x_j|)} \tag{3.9}$$

Now (3.3) follows easily from (3.9) and (3.6).

Finally to show (3.5) we only need to recall (2.3) and write

$$\begin{aligned}
&\nabla(f(R)) - \sum_{j=1}^{N} \nabla(f(R_j)) \\
= &\sum_{j=1}^{N}(f_z(R) - f_z(R_j))\nabla R_j + \sum_{j=1}^{N}(f_{\bar{z}}(R) - f_{\bar{z}}(R_j))\overline{\nabla R_j}.
\end{aligned}$$

Thanks to the above decomposition, the rest of the proof is essentially a repetition of that of (3.3). The only difference is that the constants will depend on the velocities v_j due to the terms ∇R_j. We omit further details. $\square$

Now we are ready to complete the

Proof of Theorem 1.7. By (3.1), we need to solve the integral equation (2.8) for η on $[0,\infty)\times\mathbb{R}^d$, with $W = R$ and $H = f_1(R) - \sum_{j=1}^N f_1(R_j)$. By Lemma 3.1, conditions (2.7) and (2.11) are satisfied. Thus, by Proposition 2.3, there exists $\eta \in C([0,\infty), H^1)$ with $\|\langle\nabla\rangle\eta\|_{S([t,\infty))}$ decaying exponentially in t. Since the soliton piece $R \in C([0,\infty), H^1)$, so is $u(t)$. $\square$

Proof of Theorem 1.9. This is similar to the proof of Theorem 1.7. We need to apply Proposition 2.4 with $W = R$ and $H = f(R) - \sum_{j=1}^N f(R_j)$. By Lemma 3.1, the condition (2.22) is satisfied. By Proposition 2.4, there exists $\eta \in C([T_0,\infty), H^1)$ with $\|\langle\nabla\rangle\eta\|_{S([t,\infty))}$ (in particular $\|\eta(t)\|_{H^1}$) decaying exponentially in t. $\square$

4 An infinite soliton train

In this section we construct an infinite soliton train solution to (1.1).

Thanks to Proposition 2.3, the proof of Theorem 1.1 is reduced to checking the regularity of the infinite soliton R_∞ and the tail estimates.

Lemma 4.1 (Regularity of R_∞)**.** *Let R_∞ be given as in* (1.3) *and recall $f_1(z) = |z|^\alpha z$. Then*

1. *There is a constant $\tilde{A}_1 > 0$ depending only on (A_ω, d, α), such that*

$$\|R_\infty(t)\|_\infty + \|R_\infty(t)\|_{r_1} + \sum_{j=1}^{\infty}(\|\tilde{R}_j(t)\|_\infty + \|\tilde{R}_j(t)\|_{r_1}) \le \tilde{A}_1, \qquad \forall\, t \ge 0, \tag{4.1}$$

$$\|f_1(R_\infty(t))\|_{\frac{\alpha+2-\epsilon_1}{\alpha+1}} + \sum_{j=1}^{\infty} \|f_1(\tilde{R}_j(t))\|_{\frac{\alpha+2-\epsilon_1}{\alpha+1}} \le \tilde{A}_1, \qquad \forall\, t \ge 0. \tag{4.2}$$

where $0 < \epsilon_1 < 1$ is a small constant depending on (r_1, α).

2. *There are constants $\tilde{c}_1 > 0$, $\tilde{c}_2 > 0$ depending only on (α, d), $C_1 > 0$, $C_2 > 0$ depending on $(\tilde{A}_1, d, \alpha)$, such that*

$$\|f_1(R_\infty(t)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t))\|_\infty \le C_1 e^{-\tilde{c}_1 v_\star t}, \qquad \forall\, t \ge 0, \tag{4.3}$$

$$\|f_1(R_\infty(t)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t))\|_{\frac{\alpha+2}{\alpha+1}} \le C_2 e^{-\tilde{c}_2 v_\star t}, \qquad \forall\, t \ge 0. \tag{4.4}$$

Proof of Lemma 4.1. The inequalities (4.1)–(4.2) are simple consequences of (1.5). The proof of the inequality (4.3) is similar to the proof of (3.3) and we sketch the modifications. By using (4.1) and (1.13) (fix $\eta < 1$), we have

$$\begin{aligned}
|f_1(R_\infty(t,x)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t,x))| &\lesssim \sum_{j=1}^{\infty} \Big| |R_\infty(t,x)|^\alpha - |\tilde{R}_j(t,x)|^\alpha \Big| \cdot |\tilde{R}_j(t,x)| \\
&\lesssim \sum_{j=1}^{\infty} |R_\infty(t,x) - \tilde{R}_j(t,x)|^{\min\{\alpha,1\}} |\tilde{R}_j(t,x)| \\
&\lesssim \sum_{j=1}^{\infty} \Big| \sum_{k \ne j} \omega_k^{\frac{1}{\alpha}} e^{-\eta\sqrt{\omega_k}|x - v_k t|} \Big|^{\min\{1,\alpha\}} \omega_j^{\frac{1}{\alpha}} e^{-\eta\sqrt{\omega_j}|x - v_j t|} \\
&\lesssim \sum_{j=1}^{\infty} \omega_j^{\frac{1}{\alpha}} \Big| \sum_{k \ne j} \omega_k^{\frac{1}{\alpha}} e^{-\eta(\sqrt{\omega_k}|x - v_k t| + \sqrt{\omega_j}|x - v_j t|)} \Big|^{\min\{1,\alpha\}}.
\end{aligned}$$

By (1.6), we have

$$\sqrt{\omega_k}|x - v_k t| + \sqrt{\omega_j}|x - v_j t| \ge v_\star t, \qquad \forall\, t \ge 0.$$

Hence (4.3) follows from the above estimate and (1.5). Finally (4.4) follows from interpolating the estimates (4.2)–(4.3). $\square$

We now complete the

Proof of Theorem 1.1. We first rewrite (1.7) as

$$\eta(t) = i \int_t^\infty e^{i(t-\tau)\Delta} \Big(f_1(R_\infty + \eta) - f_1(R_\infty) + f_1(R_\infty) - \sum_{j=1}^\infty f_1(\tilde{R}_j) \Big) d\tau.$$

We then apply Proposition 2.3 with $W = R_\infty$ and $H = f_1(R_\infty) - \sum_{j=1}^\infty f_1(\tilde{R}_j)$. By Lemma 4.1, it is easy to check that the condition (2.7) is satisfied. The theorem follows easily. □

5 Half-kinks

We conclude this paper by giving the proofs of Theorem 1.16 and Proposition 1.12.

Proof of Theorem 1.16. The proof is similar to that of Theorem 1.9. The only difference is that, due to the non-zero background, the profile KR is not in $\mathcal{C}(\mathbb{R}, H^1)$ any more but only in $\mathcal{C}(\mathbb{R}, H^1_{\text{loc}})$. □

Proof of Proposition 1.12. Assume $\omega = \omega_1$ and define $\zeta_1 := \zeta(\omega_1)$. Take any $\phi_0 \in (0, \zeta_1)$ and let ϕ be the solution to (1.18) on the maximal interval of existence I and with initial data

$$\phi(0) = \phi_0, \qquad \phi'(0) = \sqrt{\omega_1 \phi_0^2 - 2F(\phi_0)}.$$

We first prove that $\phi(x) \in (0, \zeta_1)$ for any $x \in I$. Indeed, assume on the contrary that there exists x_0 such that $\phi(x_0) = 0$ or $\phi(x_0) = \zeta_1$. From our choice of initial data for ϕ, it follows that, for any $x \in I$, ϕ satisfies the first integral identity

$$-\frac{1}{2}|\phi'(x)|^2 = F(\phi(x)) - \frac{\omega_1}{2}|\phi(x)|^2. \tag{5.1}$$

In particular, (5.1) at $x = x_0$ implies

$$\phi'(x_0) = 0.$$

However, by Cauchy-Lipschitz Theorem it follows that $\phi \equiv 0$ or $\phi \equiv \zeta_1$ on I, which enters in contradiction with $\phi_0 \in (0, \zeta_1)$. Hence for all $x \in I$ we have $\phi(x) \in (0, \zeta_1)$ which implies in particular that $I = \mathbb{R}$.

Since $\phi_0 \in (0, \zeta_1)$, we have $\phi'(0) > 0$ and by continuity $\phi'(x) > 0$ for x close to 0. We claim that in fact $\phi'(x) > 0$ on $\mathbb{R}$. Indeed, assume by contradiction that there exists x_0 such that $\phi'(x_0) = 0$. From the first integral (5.1), this implies that

$$F(\phi(x_0)) - \frac{\omega_1}{2}|\phi(x_0)|^2 = 0.$$

Therefore $\phi(x_0) = 0$ or $\phi(x_0) = \zeta_1$, but we have proved that to be impossible. Hence $\phi' > 0$ on $\mathbb{R}$.

We consider now the limits of ϕ at $\pm\infty$. Define

$$l := \lim_{x \to -\infty} \phi(x), \qquad L := \lim_{x \to +\infty} \phi(x).$$

Let us show that $l = 0$ and $L = \zeta_1$. Indeed, by (5.1), we have $F(l) - \frac{\omega_1}{2} l^2 = 0$ (indeed otherwise it would implies $|\phi'| > \delta > 0$ for x large, a contradiction with the boundedness of ϕ). Since $\phi \in (0, \zeta_1)$ and ϕ is increasing, this implies $l = 0$ and $L = \zeta_1$.

Let us now show that ϕ is unique up to translations. Assume by contradiction that there exists $\tilde{\phi} \in \mathcal{C}^2(\mathbb{R})$ solution to (1.18) satisfying the connection property (1.20). Since we claim uniqueness only up to translation, we can assume that $\phi(0) \in (0, \zeta_1)$. In addition, since we have shown that ϕ varies continuously from 0 to ζ_1, we can also assume without loss of generality that $\phi(0) = \phi_0 = \tilde{\phi}(0)$. The first integral identity for $\tilde{\phi}$ is for any $x \in \mathbb{R}$

$$\frac{1}{2}|\tilde{\phi}'(x)|^2 - \frac{\omega_1}{2}|\tilde{\phi}(x)|^2 + F(\tilde{\phi}(x)) = \frac{1}{2}|\tilde{\phi}'(0)|^2 - \frac{\omega_1}{2}|\tilde{\phi}(0)|^2 + F(\tilde{\phi}(0))$$

In particular, since $\lim_{x \to \pm\infty} \tilde{\phi}'(x) = 0$, and 0 and ζ_1 are zeros of $\zeta \to F(\zeta) - \frac{\omega}{2}\zeta^2$, we have

$$\frac{1}{2}|\tilde{\phi}'(0)|^2 = \frac{\omega_1}{2}|\tilde{\phi}(0)|^2 - F(\tilde{\phi}(0)).$$

As previously, it is not hard to see that ϕ' has a constant sign, which must be positive due to the limits of ϕ at $\pm\infty$. Therefore $\tilde{\phi}'(0) = \phi'(0)$ and the uniqueness follows from Cauchy-Lipschitz Theorem. Differentiating the equation we see that ϕ' verifies

$$-(\phi')'' + (\omega_1 - f'(\phi))\phi' = 0.$$

Since $\lim_{x \to -\infty}(\omega_1 - f'(\phi)) = \omega_1 - f'(0) > 0$ and $\lim_{x \to +\infty}(\omega_1 - f'(\phi)) = \omega_1 - f'(\zeta(\omega_1)) > 0$, (1.21) follows from classical ODE arguments. □

6 Multi-soliton up to time zero

In this section we add extra conditions to Theorem 1.9 so that the solution exists in $[0, \infty)$.

Theorem 6.1. *Consider* (1.1) *with* $f(u) = g(|u|^2)u$ *satisfying* (1.9) *and* (1.12)*. Let* R *be the same as in* (1.14) *and define* $v_\star$ *as in* (1.16)*. Suppose*

$$\bar{v} := \max_{k=1,\dots,N} |v_k| \leq M v_\star^M, \quad \textit{for some} \quad M \geq 1. \tag{6.1}$$

There exist constants $C > 0$*,* $c_1 > 0$*,* $c_2 > 0$ *and* $v_\sharp = v_\sharp(M) \gg 1$*, such that if* $v_\star > v_\sharp$*, then there is a unique solution* $u \in C([0, \infty), H^1)$ *to* (1.1) *satisfying*

$$e^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + e^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \leq C, \qquad \forall\, t \geq 0.$$

Remark 6.2. The extra condition (6.1) is satisfied for example if $v_j = \mu \tilde{v}_j$ for some fixed $\tilde{v}_j$ and μ is an increasing parameter.

Sketch of proof. Following the proof of Lemma 3.1, the assumption (2.22) of Proposition 2.4 is satisfied with

$$T_0 = 1, \quad \lambda = c v_\star, \quad C_1 = C_0, \quad C_2 = C_0 \bar{v},$$

where $c = C(\alpha_1)\sqrt{\min_{j=1,\dots,N}\{\omega_j\}}$ and $C_0 = C_0(d, N, \alpha_1, \alpha_2, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N)$ are independent of $(v_j)_{j=1}^N$. The smallness condition used in the proof of Proposition 2.4 is of the form

$$e^{-c\lambda_* t}(1 + C_2) \leq \varepsilon \tag{6.2}$$

for some small $\varepsilon > 0$ independent of C_2. It can be satisfied either by fixing $\lambda_* \gg 1$ independent of C_2 and then requiring $t \geq T_0$ with $T_0 = T_0(C_2)$ large (as in the proof of Proposition 2.3), or by fixing $T_0 = 1$, using the assumption $C_2 = C_0\bar{v} \leq C_0 M v_\star^M$, and requiring $v_\star$ sufficiently large. In the latter case we get a solution $\eta(t)$ for $1 \leq t < \infty$. Since the soliton piece $R \in C([0,\infty), H^1)$ and $\|\eta(t=1)\|_{H^1}$ can be chosen sufficiently small by enlarging λ_*, we can extend $\eta(t)$ up to time $t = 0$ with $O(1)$ estimates by local existence theory in H^1. $\square$

The following result is L^2-theory for L^2-subcritical and critical nonlinearities.

Theorem 6.3. *Consider* (1.1) *with* $f(u) = g(|u|^2)u$ *satisfying* (1.9) *and* (1.12). *Further assume* $\alpha_2 \leq 2/d$. *Let* R *be the same as in* (1.14) *and define* $v_\star$ *as in* (1.16). *There exist constants* $C > 0$, $c_1 > 0$, *and* $v_\sharp \gg 1$, *such that if* $v_\star > v_\sharp$, *then there is a unique solution* $u \in C([0,\infty), L^2)$ *to* (1.1) *satisfying*

$$e^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} \leq C, \qquad \forall\, t \geq 0.$$

Sketch of proof. We will modify the first part of the proof of Proposition 2.4 which bounds $\eta = u - R$ in $S([t,\infty))$. In that part, estimates for $\nabla\eta$ is only used to bound the global nonlinear terms $\sum_{j=1,2} |\eta|^{2\alpha_j+1}$ in the dual Strichartz space $N([t,\infty))$. Suppose $\alpha_2 \leq 2/d$ and

$$\|\eta\|_{S([t,\infty))} \leq e^{-\lambda t}, \quad \forall t > 0.$$

For $m = 2\alpha_j + 1$, $r = m + 1$, and a such that $\frac{2}{a} + \frac{d}{r} = \frac{d}{2}$, we have

$$\||\eta|^m\|_{N([t,\infty))} \leq \||\eta|^m\|_{L^{a'}L^{r'}(t,\infty)} \leq \|\eta\|^m_{L^{a'm}L^{r'm}(t,\infty)},$$

where $r' = r/(r-1)$ and $a' = a/(a-1)$. Let q and b be such that

$$q = r'm, \quad \frac{2}{b} + \frac{d}{q} = \frac{d}{2}.$$

We claim that $\alpha_j \leq 2/d$ is equivalent to

$$a'm \leq b. \tag{6.3}$$

Indeed, (6.3) amounts to

$$\frac{2}{a'} \geq \frac{2m}{b} = m\left(\frac{d}{2} - \frac{d}{q}\right) = m\frac{d}{2} - \frac{d}{r'},$$

i.e.

$$m\frac{d}{2} \leq \frac{d}{r'} + \frac{2}{a'} = d + 2 - \left(\frac{d}{r} + \frac{2}{a}\right) = \frac{d}{2} + 2,$$

which is exactly $\alpha_j \leq 2/d$. Thus

$$\begin{aligned}
\||\eta|^m\|_{N([t,\infty))} &\leq \left(\int_t^\infty \|\eta(s)\|^{a'm}_{L^q} ds\right)^{1/a'} = \left(\sum_{k=0}^\infty \int_{t+k}^{t+k+1} \|\eta(s)\|^{a'm}_{L^q} ds\right)^{1/a'} \\
&\leq \left(\sum_{k=0}^\infty \left(\int_{t+k}^{t+k+1} \|\eta(s)\|^b_{L^q} ds\right)^{\frac{a'm}{b}}\right)^{1/a'} \\
&\leq \left(\sum_{k=0}^\infty \left(e^{-b\lambda(t+k)}\right)^{\frac{a'm}{b}}\right)^{1/a'} = Ce^{-m\lambda t}.
\end{aligned}$$

We have used (6.3) in the second inequality. The rest of the proof is the same as the first part of the proof for Proposition 2.4. □

The following result is valid for both L^2-subcritical and L^2-supercritical nonlinearities. Its proof extends that of Proposition 2.3.

Theorem 6.4. *Consider* (1.1) *with* $f(u) = g(|u|^2)u$ *satisfying* (1.9) *and* (1.12)*. Let* $\beta_j = 2\alpha_j$*,* $j = 1, 2$*, with* $0 < \beta_1 \le \beta_2 < \alpha_{\max}$*. Assume for* $d \ge 3$

$$\frac{\beta_2}{1+\beta_2} \le \beta_1 \le \beta_2, \qquad \text{if} \quad 0 < \beta_2 < \frac{\alpha_{\max}}{2}, \tag{6.4}$$

$$\frac{\beta_2}{\alpha_{\max}+1-\beta_2} < \beta_1 \le \beta_2, \qquad \text{if} \quad \frac{\alpha_{\max}}{2} \le \beta_2 < \alpha_{\max}, \tag{6.5}$$

and for $d = 1, 2$ *we assume* (6.4) *only. Then we can choose* r_1 *and* r_2 *such that*

$$0 \le r_1 - 2 \le \beta_1 \le \beta_2 \le r_2 - 2 < \alpha_{\max}, \tag{6.6}$$

$$r_1\beta_2 \le r_1 r_2 - r_1 - r_2 \le r_2\beta_1. \tag{6.7}$$

Let R *be the same as in* (1.14) *and define* $v_\star$ *as in* (1.16)*. For any choice of* r_1, r_2 *satisfying* (6.6)*–*(6.7)*, there exist constants* $C > 0$*,* $c_1 > 0$*, and* $v_\sharp \gg 1$*, such that if* $v_\star > v_\sharp$*, then there is a unique solution* $u = R + \eta$ *to* (1.1) *on* $[0, +\infty)$ *satisfying*

$$\|\eta(t)\|_{L^{r_1}\cap L^{r_2}} \le Ce^{-c_1 v_\star t}, \quad \forall t \ge 0. \tag{6.8}$$

Moreover,

$$\|\eta\|_{S([t,\infty))} \le Ce^{-c_1 v_\star t}, \quad \forall t \ge 0.$$

Note the first strict inequality in (6.5), compared to (6.4). See Figure 2 for the β_1-β_2 region when $d = 3$. Remark also that (6.4) and (6.5) are equivalent (when $d \ge 3$) to

$$\beta_1 \le \beta_2 \le \frac{\beta_1}{1-\beta_1}, \qquad \text{if} \quad 0 < \beta_1 < \frac{\alpha_{\max}}{\alpha_{\max}+2}, \tag{6.9}$$

$$\beta_1 \le \beta_2 < \frac{(\alpha_{\max}+1)\beta_1}{1+\beta_1}, \qquad \text{if} \quad \frac{\alpha_{\max}}{\alpha_{\max}+2} \le \beta_1 < \alpha_{\max}. \tag{6.10}$$

Sketch of proof of Theorem 6.4. For $j = 1, 2$ and $\theta_j = d(\frac{1}{2} - \frac{1}{r_j}) \in (0, 1)$, we have

$$\|\eta(t)\|_{L^{r_j}} \lesssim \int_t^\infty |t-\tau|^{-\theta_j} \sum_{k=1,2} \||\eta(\tau)|^{1+\beta_k}\|_{r_j'} \, d\tau + (\text{nice terms}),$$

where $r_j' = r_j/(r_j - 1)$. The nice terms can be estimated as in the proof of Proposition 2.3. Note that

$$\||\eta|^{1+\beta_k}\|_{r_j'} = \|\eta\|^{1+\beta_k}_{(r_j')(1+\beta_k)}$$

can be estimated by Hölder inequality and (6.8) if

$$r_1 \le \frac{r_j}{r_j - 1}(1 + \beta_k) \le r_2, \quad \forall j, k. \tag{6.11}$$

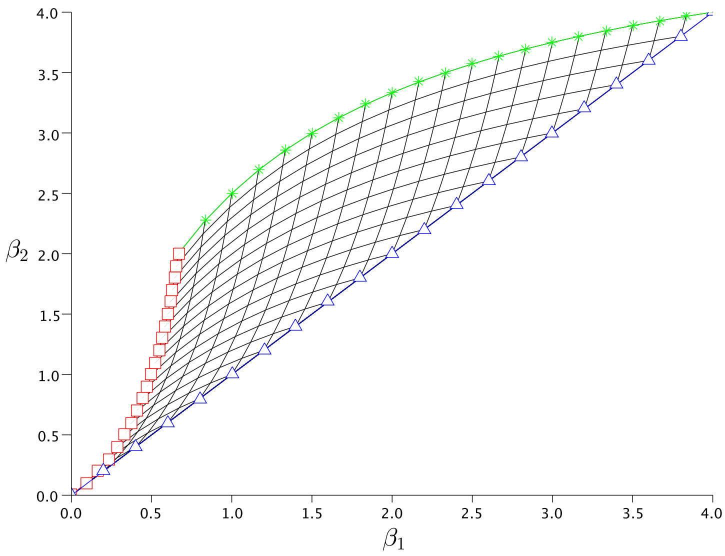

Figure 2: Region of admissible β_1, β_2 in Theorem 6.4 for $d = 3$

For $j = 1$, the left inequality of (6.11) is always true. The right inequality is equivalent to $r_1(1 + \beta_2) \le r_2(r_1 - 1)$, or

$$r_2 \le r_1(r_2 - 1 - \beta_2). \tag{6.12}$$

For $j = 2$, the right inequality of (6.11) is always true. The left inequality is equivalent to $r_1(r_2 - 1) \le r_2(1 + \beta_1)$, or

$$r_2(r_1 - 1 - \beta_1) \le r_1. \tag{6.13}$$

Equations (6.12) and (6.13) are equivalent to (6.7). Furthermore, (6.6) and (6.7) can be combined into the following equivalent condition

$$0 \le r_1 - 2 \le b_1(r_1, r_2) \le \beta_1 \le \beta_2 \le b_2(r_1, r_2) \le r_2 - 2 < \alpha_{\max} \tag{6.14}$$

where

$$b_1(r_1, r_2) = r_1 - 1 - r_1/r_2, \qquad b_2(r_1, r_2) = r_2 - 1 - r_2/r_1.$$

It turns out that when $2 \le r_1 \le r_2 < \alpha_{\max} + 2$ we always have

$$0 \le r_1 - 2 \le b_1(r_1, r_2) \le b_2(r_1, r_2) \le r_2 - 2 < \alpha_{\max}.$$

Thus for any (β_1, β_2) in the right triangle with a vertex $(b_1(r_1, r_2), b_2(r_1, r_2))$ and hypotenuse on the line $\beta_1 = \beta_2$, the pair r_1, r_2 satisfies (6.6) and (6.7).

Denote the curve $\Gamma(r_1)$ for fixed $2 \le r_1 < 2 + \alpha_{\max}$,

$$\Gamma(r_1) = \{(b_1(r_1, r_2), b_2(r_1, r_2)) : r_1 \le r_2 \le 2 + \alpha_{\max}\}.$$

It satisfies

$$b_2 = \frac{b_1}{r_1 - 1 - b_1}, \quad b_1 = \frac{(r_1 - 1)b_2}{1 + b_2},$$

and starts at (r_1-2, r_1-2). It goes to infinity with asymptote $b_1 = r_1-1$ for $d=1,2$, while ends at $\Sigma(2+\alpha_{\max})$ to be defined below for $d \geq 3$. It moves to the right as r_1 increases.

Denote the curve $\Sigma(r_2)$ for fixed $2 < r_2 \leq 2+\alpha_{\max}$,

$$\Sigma(r_2) = \{(b_1(r_1,r_2), b_2(r_1,r_2)) : 2 \leq r_1 \leq r_2\}.$$

It satisfies

$$b_2 = (r_2-1)\frac{b_1}{1+b_1}.$$

It starts at $\Gamma(2)$ and ends at (r_2-2, r_2-2). It moves upward as r_2 increases.

For given $0 < \beta_1 < \beta_2 < \alpha_{\max}$, conditions (6.4)–(6.5) imply that (β_1,β_2) is on the right of $\Gamma(2)$ and, if $d \geq 3$, is below $\Sigma(\alpha_{\max})$. Thus we can find $R_1 = R_1(\beta_1,\beta_2)$ and $R_2 = R_2(\beta_1,\beta_2)$ such that (β_1,β_2) is the intersection point of $\Gamma(R_1)$ and $\Sigma(R_2)$, and $R_1 \leq R_2$. To satisfy (6.14), we can either choose $(r_1,r_2) = (R_1,R_2)$, or any $2 \leq r_1 < R_1 \leq R_2 < r_2 < 2+\alpha_{\max}$ as long as the intersection point $\Gamma(r_1) \cap \Sigma(r_2)$ is at upper-left direction to (β_1,β_2).

The above shows we can estimate $|\eta|^{1+\beta_k}$ in $L^{r_j'}$ for $j,k=1,2$.

For the Strichartz estimate, since $(2/\theta_1, r_1)$ is admissible, we have with $a = (2/\theta_1)'$

$$\begin{aligned}\|\eta\|_{S([t,\infty))} &\lesssim \|f(W+\eta) - f(W) + H\|_{L^a(t,\infty;L^{r_1'})}\\ &\lesssim \|e^{-c_1 v_* \tau}\|_{L^a(t,\infty)} \lesssim v_*^{-1/a} e^{-c_1 v_* t}.\end{aligned}$$

□

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